GRAPH SIGNATURE ARITHMETIC
A FIRST-PRINCIPLES ARITHMETIC OF FINITE GRAPHS

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Abstract. Graph signature arithmetic is an arithmetic on finite simple graphs built from two counts: vertices and edges. A graph G has signature $\sigma(G) = (v(G), e(G))$ and score $s(G) = v(G) + e(G)$. Two operations make this arithmetic nontrivial. The first is fusion of isolated numerals, which turns m and n into the complete bipartite graph $K_{m,n}$ and gives the law $s(m \oplus n) = m + n + mn$. The second is the lexicographic product $O[R]$, read directionally: O is the organizer and R is the payload. Its exact score law is

$$s(O[R]) = v(O)s(R) + e(O)v(R)^2.$$

For fixed payload R, product scores fall into arithmetic lanes

$$L_R(\beta; t) = \mathrm{Step}(R)t + v(R)^2(\beta - 1), \qquad \mathrm{Step}(R) = v(R) + e(R) + v(R)^2,$$

where $t = v(O)$ and $\beta = \beta_1(O)$ is the cycle rank of the organizer. The payload is globally prime-dead exactly when $\gcd(v(R), e(R)) > 1$; otherwise at least one lane is live, and Dirichlet's theorem gives infinitely many prime scores in that lane.

The score layer is deliberately blind: it depends on the payload only through (v, e). Therefore nonisomorphic payloads with the same signature cast the same arithmetic shadow, even when one is planar and another is forced nonplanar. Crossing number and quotient recovery are then introduced only as honest refinements, not as hidden assumptions. The paper keeps the exact core separate from topological conjectures. What remains is a small, rigorous arithmetic of graph signatures: elementary, directional, prime-bearing, and visibly incomplete in exactly the places where topology enters.

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1. The seed

A structure is not only a set of things. It is also the pattern of relations among those things. For a finite graph, that gives two primitive counts:

$$v(G) = |V(G)|, \qquad e(G) = |E(G)|.$$

The first scalar attached to the graph is their sum:

$$s(G) = v(G) + e(G).$$

This paper studies what happens when that very small measurement is allowed to interact with graph operations.

The point is not to claim that $s(G)$ sees everything. It does not. In fact, one of the main lessons is that it forgets most topology. The point is sharper: once a graph is reduced to the signature

$$\sigma(G) = (v(G), e(G)),$$

there is still a real arithmetic left behind. Product scores fall into explicit progressions. Prime-bearing and prime-dead behavior is controlled by one gcd. Topologically different graphs can become arithmetically indistinguishable. That blindness is not a bug; it is the exact boundary of the first layer.

Remark 1.1 (What has been purified). Earlier drafts mixed this score arithmetic with crossing theory, dessins d'enfants, Belyi maps, modular forms, Galois orbits, and analytic-number-theoretic analogies. Those themes are interesting, but they are not needed for the first theorem layer. This paper strips the system to the part that follows from finite graph counts, complete bipartite bridges, the lexicographic product, and Dirichlet's theorem on arithmetic progressions.

2. Objects and observables

Definition 2.1 (Graphs)**.** All graphs in this paper are finite simple graphs, taken up to isomorphism. A graph may be disconnected unless connectedness is explicitly assumed.

Definition 2.2 (Signature and score)**.** For a graph G, define

$$v(G) = |V(G)|, \qquad e(G) = |E(G)|, \qquad s(G) = v(G) + e(G).$$

The pair

$$\sigma(G) = (v(G), e(G))$$

is called the *signature* of G. The scalar $s(G)$ is called the *score*.

Definition 2.3 (Components and cycle rank)**.** Let $\beta_0(G)$ be the number of connected components of G. Define the cycle rank

$$\beta_1(G) = e(G) - v(G) + \beta_0(G).$$

When G is connected,

$$\beta_1(G) = e(G) - v(G) + 1.$$

Remark 2.4 (The first honesty line). The signature $\sigma(G) = (v, e)$ does not determine G. It does not determine degree sequence, planarity, crossing number, automorphism group, chromatic number, or factorization. It is intentionally sparse. The arithmetic below is the arithmetic of what survives that forgetting.

3. Isolated numerals and fusion

Definition 3.1 (Isolated numeral)**.** For $n \geq 0$, the isolated numeral n is the graph on n isolated vertices and no edges. Thus

$$v(n) = n, \qquad e(n) = 0, \qquad s(n) = n.$$

Definition 3.2 (Fusion)**.** For isolated numerals m and n, define

$$m \oplus n$$

to be the graph obtained by keeping all $m+n$ vertices and adding every edge between the left block of size m and the right block of size n. Equivalently,

$$m \oplus n \cong K_{m,n}.$$

Theorem 3.3 (Fusion law)**.** *For isolated numerals* $m, n \geq 0$*,*

$$v(m \oplus n) = m + n, \qquad e(m \oplus n) = mn,$$

and therefore

$$\boxed{s(m \oplus n) = m + n + mn = (m+1)(n+1) - 1.}$$

Proof. The fused graph is $K_{m,n}$, which has $m+n$ vertices and mn edges. Add the two counts. ☐

Proposition 3.4 (Fusion planarity threshold)**.** *The fusion* $m \oplus n \cong K_{m,n}$ *is planar if and only if*

$$\min(m, n) \leq 2.$$

In particular, the first forced nonplanar fusion is

$$3 \oplus 3 \cong K_{3,3}.$$

Proof. The complete bipartite graph $K_{m,n}$ is planar exactly when one part has size at most 2. This is the standard planarity criterion for complete bipartite graphs; $K_{3,3}$ is the first forbidden case. ☐

Remark 3.5 (The first phase change)*.* Fusion begins as ordinary addition of isolated dots. The first new edge appears as a relation between the two blocks. The first forced topological obstruction appears at $3 \oplus 3$. The score law is elementary, but topology already waits one layer above it.

4. The structural product

The second operation is a standard graph operation, but read with direction. In graph theory it is the lexicographic product. In this arithmetic, the left factor organizes and the right factor supplies the payload.

Definition 4.1 (Structural product)**.** Let O and R be connected graphs. The structural product

$$O \star R$$

is the graph formed as follows:

(i) replace each vertex of O by a full copy of R;
(ii) for every edge of O, join the corresponding two copies of R by a complete bipartite bridge.

Equivalently, $O \star R$ is the lexicographic product $O[R]$. We call O the *organizer* and R the *payload.*

Theorem 4.2 (Exact product laws)**.** *Let O and R be connected graphs and put $X = O \star R$. Then*

$$\boxed{v(X) = v(O)v(R),}$$

$$\boxed{e(X) = v(O)e(R) + e(O)v(R)^2,}$$

and therefore

$$\boxed{s(X) = v(O)s(R) + e(O)v(R)^2.}$$

Proof. There is one copy of R above each vertex of O, so the vertex count is $v(O)v(R)$. The internal edges are the edges inside those copies, giving $v(O)e(R)$. Each organizer edge creates a complete bipartite bridge between two copies of R, with $v(R)^2$ bridge edges. Summing over the $e(O)$ organizer edges gives the edge law. The score law follows by adding vertices and edges. ☐

Theorem 4.3 (Degree transport)**.** *Let (x, y) be the vertex of $O \star R$ lying over $x \in V(O)$ and corresponding to $y \in V(R)$. Then*

$$\boxed{\deg_{O\star R}(x, y) = \deg_R(y) + v(R)\deg_O(x).}$$

Proof. The vertex (x, y) has internal neighbors inside the copy of R over x, contributing $\deg_R(y)$. For each neighbor of x in O, the complete bipartite bridge connects (x, y) to all $v(R)$ vertices in the adjacent copy. There are $\deg_O(x)$ such organizer neighbors. ☐

Proposition 4.4 (Basic algebra)**.** *For connected graphs:*

(i) $K_1 \star R \cong R$;
(ii) $(O \star R) \star H \cong O \star (R \star H)$;
(iii) $O \star R$ *is not generally isomorphic to* $R \star O$.

Proof. These are the standard identity, associativity, and noncommutativity properties of the lexicographic product. The identity statement is immediate. Associativity follows by comparing the triples of coordinates and the adjacency rule. Noncommutativity is witnessed by examples with different degree behavior, for instance $K_2 \star P_3$ and $P_3 \star K_2$. ☐

Proposition 4.5 (Clique arithmetic)**.** *For complete graphs,*

$$\boxed{K_m \star K_n \cong K_{mn}.}$$

Proof. Each copy of K_n is complete, and every pair of organizer vertices in K_m is adjacent, so every pair of vertices in the product is adjacent. ☐

5. Multiplicative primitiveness

Definition 5.1 (Nontrivial factor)**.** A connected graph is called nontrivial if it is not K_1.

Definition 5.2 (Multiplicatively primitive graph)**.** A connected graph G is *multiplicatively primitive* if there are no nontrivial connected graphs O and R such that

$$G \cong O \star R.$$

Proposition 5.3 (Vertex-product obstruction)**.** *If*

$$G \cong O \star R$$

with O and R both nontrivial and connected, then

$$v(G) = v(O)v(R)$$

with both factors at least 2*. In particular, every connected graph with prime vertex count is multiplicatively primitive.*

Proof. The vertex law gives $v(G) = v(O)v(R)$. Nontrivial connected factors have at least two vertices. If $v(G)$ is prime, no such factorization of the vertex count is possible. ☐

Proposition 5.4 (Trees are primitive)**.** *Every nontrivial tree is multiplicatively primitive.*

Proof. Suppose $T \cong O \star R$ with O, R nontrivial and connected. Since O is connected and nontrivial, it has an edge. Since R is connected and nontrivial, $v(R) \geq 2$. The bridge over any organizer edge contains a $K_{v(R),v(R)}$, hence contains a cycle. Therefore $O \star R$ cannot be a tree. ☐

Proposition 5.5 (Composite clique family)**.** *If $N = mn$ with $m, n \geq 2$, then*

$$K_N \cong K_m \star K_n.$$

Thus every complete graph with composite vertex count is composite under $\star$.

Remark 5.6 (Primitive does not mean prime score). Multiplicative primitiveness is a graph-factorization property. A graph can be primitive while its score is composite, and a graph can have prime score while not being primitive. The score arithmetic and the product arithmetic touch, but they are not the same object.

6. Arithmetic lanes

Now fix a connected payload R. The score law turns all products $O \star R$ into arithmetic progressions.

Definition 6.1 (Payload step)**.** For a connected graph R, define

$$\boxed{\mathrm{Step}(R) = v(R) + e(R) + v(R)^2 = s(R) + v(R)^2.}$$

Definition 6.2 (Lane)**.** Let R be connected. For an integer $\beta \geq 0$ and an integer $t \geq 1$, define

$$\boxed{L_R(\beta; t) = \mathrm{Step}(R)t + v(R)^2(\beta - 1).}$$

When $\beta = 0$, we write

$$L_R^{\mathrm{tree}}(t) = \mathrm{Step}(R)t - v(R)^2$$

and call it the *tree lane.*

Theorem 6.3 (Lane formula)**.** *Let O and R be connected, and set*

$$t = v(O), \qquad \beta = \beta_1(O).$$

Then

$$\boxed{s(O \star R) = L_R(\beta; t).}$$

Proof. Since O is connected,

$$e(O) = v(O) - 1 + \beta_1(O) = t - 1 + \beta.$$

Using the score law,

$$\begin{aligned} s(O \star R) &= t\, s(R) + (t - 1 + \beta)v(R)^2 \\ &= t\big(v(R) + e(R) + v(R)^2\big) + v(R)^2(\beta - 1) \\ &= \mathrm{Step}(R)t + v(R)^2(\beta - 1). \end{aligned}$$

□

Remark 6.4 (What the lane formula says). The payload fixes the step size. The organizer cycle rank chooses the lane. The organizer vertex count moves along the lane. This is the central arithmetic mechanism.

Lemma 6.5 (Realizing fixed cycle rank)**.** *For every fixed $\beta \geq 0$, there exists T_β such that for every $t \geq T_\beta$ there is a connected simple graph O with*

$$v(O) = t, \qquad \beta_1(O) = \beta.$$

Proof. For $\beta = 0$, take any tree on t vertices. For $\beta > 0$, begin with a tree on t vertices and add β distinct non-tree edges. This is possible once the complete graph has at least β more edges than a tree, i.e.

$$\binom{t}{2} - (t - 1) \geq \beta.$$

That inequality holds for all sufficiently large t. □

Corollary 6.6 (Lanes are eventually realized progressions)**.** *For fixed connected R and fixed $\beta \geq 0$, the numerical values $L_R(\beta; t)$ form an arithmetic progression modulo* $\mathrm{Step}(R)$*, and all sufficiently large values in that lane are realized by actual organizers of cycle rank β.*

7. The global death law

Definition 7.1 (Live and dead lanes)**.** The lane $L_R(\beta; t) = qt + a$ is called *live* if

$$\gcd(q, a) = 1.$$

It is called *dead* if $\gcd(q, a) > 1$. A dead lane contains no infinite family of primes; every term is divisible by the same integer > 1, apart from the possibility of a finite exceptional term equal to that divisor.

Definition 7.2 (Global prime-death)**.** A connected payload R is *globally prime-dead* if every lane $L_R(\beta; t)$ is dead. It is *globally prime-viable* if at least one lane is live.

Theorem 7.3 (Global death law)**.** *Let R be connected. Then*

$$\boxed{R \textit{ is globally prime-dead} \iff \gcd(v(R), e(R)) > 1.}$$

Equivalently,

$$\boxed{R \textit{ is globally prime-viable} \iff \gcd(v(R), e(R)) = 1.}$$

Proof. Write

$$a = v(R), \qquad b = e(R).$$

Then

$$\mathrm{Step}(R) = a + b + a^2.$$

The common obstruction across all lanes is

$$\begin{aligned} \gcd(\mathrm{Step}(R), a^2) &= \gcd(a + b + a^2, a^2) \\ &= \gcd(a + b, a^2). \end{aligned}$$

This gcd is greater than 1 exactly when some prime divides both a and b. Indeed, if $p \mid a$ and $p \mid b$, then $p \mid a + b$ and $p \mid a^2$. Conversely, if $p \mid a^2$ and $p \mid a + b$, then $p \mid a$, hence $p \mid b = (a + b) - a$.

If $\gcd(a, b) > 1$, then every intercept $a^2(\beta - 1)$ and the step $\mathrm{Step}(R)$ share a common divisor > 1, so every lane is dead.

If $\gcd(a, b) = 1$, then

$$\gcd(\mathrm{Step}(R), a^2) = 1.$$

In particular the tree lane

$$L_R^{\mathrm{tree}}(t) = \mathrm{Step}(R)t - a^2$$

is live. Thus R is not globally dead. ☐

Corollary 7.4 (Tree immunity)**.** *Every tree payload is globally prime-viable.*

Proof. If R is a tree, then $e(R) = v(R) - 1$, so

$$\gcd(v(R), e(R)) = \gcd(v(R), v(R) - 1) = 1.$$

☐

Corollary 7.5 (Unicyclic death)**.** *Every connected unicyclic payload with $v(R) \geq 3$ is globally prime-dead.*

Proof. For a connected unicyclic graph, $e(R) = v(R)$, so $\gcd(v(R), e(R)) = v(R) > 1$. ☐

Corollary 7.6 (Complete graph death)**.** *For $n \geq 3$, the complete graph K_n is globally prime-dead.*

Proof. Here

$$v(K_n) = n, \qquad e(K_n) = \binom{n}{2} = \frac{n(n-1)}{2}.$$

If n is odd, then n divides $e(K_n)$. If n is even and $n \geq 4$, then $n/2$ divides both n and $e(K_n)$. The case $n = 3$ is included in the odd case. □

Definition 7.7 (Prime-fertile and live-composite)**.** A connected payload R is called *prime-fertile* if

$$\gcd(v(R), e(R)) = 1 \qquad \text{and} \qquad \mathrm{Step}(R) \text{ is prime.}$$

It is called *live-composite* if $\gcd(v(R), e(R)) = 1$ but $\mathrm{Step}(R)$ is composite.

Proposition 7.8 (Tree fertility sequence)**.** *If R is a tree with $n = v(R)$, then*

$$\mathrm{Step}(R) = n + (n-1) + n^2 = (n+1)^2 - 2.$$

Thus a tree payload is prime-fertile exactly when $(n+1)^2 - 2$ is prime.

8. The Dirichlet bridge

The word "prime" in this paper means prime score values in the numerical lanes. The bridge to classical primes is exactly Dirichlet's theorem.

Theorem 8.1 (Dirichlet bridge)**.** *Let R be a connected payload and let $\beta \geq 0$. If the lane*

$$L_R(\beta; t) = \mathrm{Step}(R)t + v(R)^2(\beta - 1)$$

is live, then it contains infinitely many prime numbers. Moreover, for all sufficiently large such prime values, there are connected organizers O with $v(O) = t$ and $\beta_1(O) = \beta$ realizing them as scores $s(O \star R)$.

Proof. A live lane is an arithmetic progression

$$qt + a, \qquad q = \mathrm{Step}(R), \qquad a = v(R)^2(\beta - 1),$$

with $\gcd(q, a) = 1$. By Dirichlet's theorem, this progression contains infinitely many primes. By the realization lemma, every sufficiently large t occurs as the vertex count of a connected organizer with cycle rank β, so the corresponding sufficiently large prime lane values are realized as product scores. □

Corollary 8.2 (Canonical prime lane)**.** *If $\gcd(v(R), e(R)) = 1$, then the tree lane*

$$L_R^{\mathrm{tree}}(t) = \mathrm{Step}(R)t - v(R)^2$$

contains infinitely many prime scores.

Remark 8.3 (What Dirichlet does and does not say)*.* Dirichlet supplies prime values in live numerical progressions. It does not say those primes are graph primes under $\star$. It says the score sequence generated by a fixed payload and organizer family hits ordinary prime numbers infinitely often.

9. Arithmetic shadows

Theorem 9.1 (Arithmetic shadow)**.** *For fixed organizer parameters t and β, the product score $s(O \star R)$ depends on the payload R only through its signature*

$$\sigma(R) = (v(R), e(R)).$$

Consequently, if R_1 and R_2 have the same signature, then they generate the same lane formulas and the same global death/live classification.

Proof. The lane formula uses R only through $v(R)$ and $e(R)$:

$$s(O \star R) = (v(R) + e(R) + v(R)^2)t + v(R)^2(\beta - 1).$$

The global death law also uses only $\gcd(v(R), e(R))$. ☐

Remark 9.2 (The sparse layer). This is the central blindness theorem. The score arithmetic can distinguish signatures, but it cannot distinguish nonisomorphic graphs with the same signature. Any richer geometry must be added as a second layer.

Example 9.3 (Twin payloads)**.** The path P_4 and the star $K_{1,3}$ are nonisomorphic, but both have

$$(v, e) = (4, 3).$$

Therefore

$$\mathrm{Step} = 4 + 3 + 16 = 23,$$

and every lane attached to P_4 is also the corresponding lane attached to $K_{1,3}$. Signature arithmetic cannot see the difference between them.

Example 9.4 (A nonplanar live payload)**.** Let R be any connected graph with

$$v(R) = 6, \qquad e(R) = 13.$$

Such a graph is forced nonplanar by Euler's planar bound, since a simple planar graph with $v = 6$ has at most

$$3v - 6 = 12$$

edges. But

$$\gcd(6, 13) = 1,$$

so R is globally prime-viable. Its tree lane is

$$L_R^{\mathrm{tree}}(t) = (6 + 13 + 36)t - 36 = 55t - 36,$$

which is live because $\gcd(55, 36) = 1$.

Example 9.5 (A stronger nonplanar live payload)**.** If

$$v(R) = 7, \qquad e(R) = 17,$$

then R is forced nonplanar because

$$17 > 3 \cdot 7 - 6 = 15,$$

yet

$$\gcd(7, 17) = 1.$$

Here

$$\mathrm{Step}(R) = 7 + 17 + 49 = 73,$$

which is prime, and the tree lane is

$$73t - 49.$$

Remark 9.6 (The lesson of the monsters). Planarity and prime-viability are independent filters. A payload may be topologically complicated and still arithmetically live. The signature layer does not measure flatness; it measures the arithmetic residue left by (v, e).

10. Topology as a second layer

The first layer is not wrong because it forgets topology. It is useful because we know exactly what it forgets.

Definition 10.1 (Crossing number)**.** Let $c(G)$ be the crossing number of G, the minimum number of edge crossings over all planar drawings of G.

Definition 10.2 (Extended signature)**.** The extended signature is

$$\sigma_x(G) = (v(G), e(G), c(G)).$$

This is not the full topology of G, but it is the first correction to the arithmetic shadow.

Proposition 10.3 (Quotient recovery with given blocks)**.** *Let $X = O \star R$, and let the product block partition be given. Collapsing each copy of R to a single vertex recovers the organizer:*

$$X/R \cong O.$$

Proof. There is one block for each vertex of O. Between two distinct blocks, the product has all possible cross-edges exactly when the corresponding organizer vertices are adjacent, and no cross-edges otherwise. Collapsing blocks therefore recovers the adjacency relation of O. ☐

Definition 10.4 (Rigid factorization)**.** A factorization $X \cong O \star R$ is called *rigid* if the payload block partition is unique and both quotient directions are available:

$$X/R \cong O, \qquad O \backslash X \cong R.$$

Proposition 10.5 (Directional topological recovery)**.** *In a rigid factorization $X \cong O \star R$,*

$$c(X/R) = c(O), \qquad c(O \backslash X) = c(R).$$

Proof. By rigidity, the right quotient is isomorphic to O and the left quotient is isomorphic to R. Crossing number is an isomorphism invariant. ☐

Remark 10.6 (No hidden uniqueness claim). This paper does not claim that every product admits unique factorization, nor that the payload can always be recovered from an unlabeled product. Quotient recovery is exact when the factorization and block structure are part of the data, and directional topological recovery is exact in the rigid regime. Classifying that regime is a separate problem.

11. Crossing growth and bridge pressure

The product $O \star R$ is structurally quadratic in payload size, because every organizer edge creates a complete bipartite bridge.

Proposition 11.1 (Quadratic edge growth)**.** *Fix a connected organizer O and let R vary over connected payloads with $v(R) = n$. Then*

$$e(O \star R) = v(O)e(R) + e(O)n^2.$$

In particular, the bridge part grows like $O(n^2)$.

Definition 11.2 (Zarankiewicz number)**.** Let

$$Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

This is the classical Zarankiewicz expression for the conjectural crossing number of $K_{m,n}$.

Conjecture 11.3 (Crossing transport)**.** *For connected O and R,*

$$c(O \star R) \geq v(O)c(R) + e(O)Z(v(R), v(R)).$$

Remark 11.4 (Why this is conjectural). The inequality says that every payload copy contributes its own crossing cost and every organizer bridge contributes at least the complete bipartite crossing pressure of $K_{v(R),v(R)}$. This is geometrically natural, but not proved here. It is intentionally stated as a conjecture.

Proposition 11.5 (Quadratic versus quartic)**.** *Assuming crossing transport, for fixed organizer O and payload size $n = v(R)$, the bridge edge contribution grows like $O(n^2)$ while the bridge crossing lower bound grows like $O(n^4)$.*

Proof. The bridge edge term is $e(O)n^2$. The Zarankiewicz expression satisfies

$$Z(n,n) \sim \frac{n^4}{16}.$$

$\square$

12. Open problems

Problem 12.1. Classify the connected graphs that are multiplicatively primitive under $\star$.

Problem 12.2. Classify rigid factorizations. Given a graph X, when is there a unique factorization $X \cong O \star R$?

Problem 12.3. For fixed signature (a, b) with $\gcd(a, b) = 1$, classify the possible topological types of connected payloads. How large can crossing number become inside one live arithmetic shadow class?

Problem 12.4. For a fixed step q, classify all payload signatures (a, b) satisfying

$$a + b + a^2 = q.$$

Which steps arise from globally viable payloads?

Problem 12.5. Prove or disprove the crossing transport conjecture:

$$c(O \star R) \geq v(O)c(R) + e(O)Z(v(R), v(R)).$$

Problem 12.6. Develop a true second-layer topology for signature arithmetic. Which graph invariants refine the arithmetic shadow without destroying the lane structure?

13. Conclusion

Graph signature arithmetic begins with two counts:

$$v(G), \qquad e(G).$$

Their sum is crude, but not empty. Under complete bipartite fusion it obeys

$$s(m \oplus n) = (m + 1)(n + 1) - 1.$$

Under the structural product $O \star R$, it obeys

$$s(O \star R) = v(O)s(R) + e(O)v(R)^2.$$

From that one law, the lane formula follows:

$$s(O \star R) = \mathrm{Step}(R)v(O) + v(R)^2(\beta_1(O) - 1).$$

And from that lane formula, the global death law follows:

$$R \text{ is globally prime-dead} \iff \gcd(v(R), e(R)) > 1.$$

That is the purified core. The payload's signature determines its arithmetic shadow. Dirichlet's theorem supplies infinitely many primes in every live lane. But topology is not contained in the signature: nonisomorphic, even topologically different, graphs can cast the same arithmetic shadow. The first layer produces arithmetic; the second layer must recover shape.

The restriction is deliberate. Score primes are not graph-theoretic prime factors, and this paper does not claim unique factorization under $\star$, canonical quotient recovery for all products, automatic dessin structure, modularity, Belyi correspondence, Galois orbit classification, or any zeta-zero consequence. Those larger geometric and analytic ideas may return as later layers, but only when their hypotheses are stated as cleanly as the hypotheses above.

So the paper stops at the right place. The exact score arithmetic is complete: fusion, structural multiplication, score lanes, the gcd death law, Dirichlet prime infinitude in live realized lanes, arithmetic shadow blindness, and quotient recovery under explicit rigid-factorization hypotheses. Everything beyond that is future structure, not hidden theorem.

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