# A CRITICAL HARDY-SPACE CONTRACTION FOR THE CUBIC CHEBYSHEV TRACE

BLAIZE ROUYEA AND COREY BOURGEOIS

Abstract. We continue the cubic descent and Chebyshev trace program of the authors. The earlier work produces a finite-local trace $(u_n)$ whose boundedness is equivalent to the Riemann Hypothesis. The present paper does not prove RH. Instead, it proves an unconditional contraction theorem for the exact prime-filter hierarchy and isolates the arithmetic obstruction that remains.

The critical Cayley coordinate

$$\chi = \frac{s-2}{s+1}$$

maps the critical line to $|\chi| = 1$. In this coordinate the singular Chebyshev branch factors through an explicit Schur function $\phi$ with $\phi(0) = 16/27$. The trace filters admit an exact Laguerre–Hardy representation, from which we prove

$$\sup_{n\geq 0} \int_0^\infty e^{-3x} |\Pi_n(x)|^2 \, dx < \infty.$$

The first two weighted critical moments are then $O(n)$ and $O(n^2)$, and the prime-square quadratic variation is polynomially bounded.

A complementary $S_3$ Fourier calculation gives the exact identity

$$B_0(1 - \beta_1\beta_2) = 6$$

and identifies the normalized standard-channel Green defect with the Chebyshev–Krein $U$-port. On the real invariant axis the completed source has a rank-one negative future component. The completed endpoint residue is then represented by a coefficient of $\mathcal{H}G\phi^{n-1}$, where all filter growth is controlled and every possible interior exponential instability is carried by a pole of $\xi'/\xi$.

An explicit symmetric, positive, real-log-convex countermodel with off-symmetry zeros shows why the real-axis contraction cannot be promoted by a generic maximum principle. Thus the remaining pole exclusion is left explicit rather than hidden inside a Hardy-space assumption.

## 1. Introduction

The Riemann Hypothesis (RH) asserts that every nontrivial zero $\rho$ of the Riemann zeta function satisfies $\operatorname{Re} \rho = 1/2$. There are many exact reformulations of this statement: positivity of Li coefficients [5, 2], moment and Hankel criteria, Jensen-polynomial hyperbolicity [4, 7], horizontal modulus monotonicity [12], and de Branges or Hermite–Biehler-type formulations, among others. The existence of an equivalence is therefore not, by itself, evidence of progress toward RH. What matters is whether a change of coordinates removes a genuine obstruction rather than merely renaming it.

In [8], we introduced an order-three Möbius descent of the Riemann xi function. The descended entire function $K$ has order $1/2$ and genus zero; the critical line becomes a negative real ray, and RH becomes a one-tower moment problem. In [9], we then made a lossless Chebyshev change of basis and obtained a trace $(u_n)$ for which RH is equivalent to boundedness, or equivalently to the absence of exponential Green-function escape. A companion calculation [10] studies the

2020 *Mathematics Subject Classification.* Primary 11M26; Secondary 30H10, 33C45, 42C05, 47B35.

*Key words and phrases.* Riemann hypothesis, Riemann xi function, Hardy space, Schur function, Chebyshev polynomial, Laguerre polynomial, cubic descent, Krein space, prime filter.

residue-first boundary layer near $s = 2$; it is cited here for context only and is not used as an input to the main Hardy contraction theorem.

Hilbert-space and positive-kernel formulations of RH have substantial antecedents, including Li–Bombieri–Lagarias criteria and the screw-function/Weil framework of Suzuki [5, 2, 11]. The present construction is different in scope: it concerns the exact finite-local prime filters attached to the cubic trace, and the new theorem is an unconditional norm bound for that filter family rather than a new positivity equivalence.

The present paper begins where those three constructions meet.

**Architecture of the argument.** The argument is organized in four layers:

$$\text{Krein low/high block} \longrightarrow S_3 \text{ current compression}$$
$$\longrightarrow \text{critical Schur–Hardy contraction} \longrightarrow \text{completed arithmetic source.}$$

The first layer records the exponential signal of an off-line zero. The middle two layers are unconditional and remove geometric amplification. The last layer is where the arithmetic pole set of $\xi'/\xi$ remains visible.

The principal unconditional theorem of this paper is the third layer. It proves the critical second-moment estimate that was only numerical in the earlier trace analysis. The first two weighted moments and the prime-square sampling bound then follow by elementary Laguerre operator estimates. The $S_3$ calculation is complementary: it shows that an apparently dangerous $z^2$ amplification was a normalization artifact, and it identifies the exact Krein port in which the completed second moment lives.

We emphasize the logical boundary. The real-axis $S_3$ contraction does not automatically pass through the residue projection, and the critical Hardy boundary is not the real invariant axis. Likewise, the completed source $\mathcal{H}$ belongs to $H^2(\mathbb{D})$ only if the very poles excluded by RH are absent. These facts are stated explicitly below to prevent a false closure by positivity or Cauchy–Schwarz.

**What is new here.** The results not contained in [8, 9, 10] are:

(1) the $S_3$ determinant identities $B_0(1 - \beta_1\beta_2) = 6$ and
$$\frac{W^2}{1 - \beta_1\beta_2} = -\frac{9}{32}(1 - y^2),$$
which identify the normalized standard-channel Green defect with the complementary Chebyshev–Krein port;

(2) an unconditional rank-one source contraction on the real invariant axis;

(3) the critical Cayley–Schur factorization and the exact coefficient formula
$$b_{n,k} = -\frac{3}{8}[\chi^{2n+1-k}]G(\chi)\phi(\chi)^{n-1};$$

(4) the uniform critical filter estimate
$$\int_0^\infty e^{-3x}|\Pi_n(x)|^2\,dx = O(1),$$
together with the $O(n)$ and $O(n^2)$ weighted-moment bounds;

(5) a polynomial discrete sampling theorem implying polynomial prime-square quadratic variation;

(6) an exact completed-source coefficient identity that isolates the remaining pole obstruction.

We do not claim that these results provide a complete proof of RH, nor do we attempt to rank inequivalent RH criteria by a notion of "distance to proof." The novelty claimed is narrower and verifiable: a previously uncontrolled critical filter hierarchy is shown to be uniformly contractive in a natural Hardy space, and the remaining exponential instability is localized to the arithmetic pole set itself.

## 2. Minimal setup from the cubic descent and trace

We state all algebraic data used in the present proofs. The descent and the RH-equivalent trace criterion are proved in the publicly available companion papers [8, 9]; no result from the boundary-layer note is needed below.

Let

$$w(s) = \frac{s-1}{s}, \qquad w^3 = \mathrm{id},$$

and define the cubic invariant

$$q(s) = \frac{(s-2)(s+1)(2s-1)}{2s(s-1)}, \qquad c = \frac{27}{4}. \tag{1}$$

The identity

$$q(s)^2 + c = \frac{(s^2-s+1)^3}{s^2(s-1)^2} \tag{2}$$

will be used repeatedly. Put

$$\ell(s) = \frac{\xi'(s)}{\xi(s)}, \qquad W(s) = \frac{c}{q(s)^2}, \qquad y(s) = 1 + 2W(s).$$

The trace prime filters are

$$\Pi_n(x) = -\operatorname*{Res}_{s=2} e^{-x(s-2)} W(s) T_n(1+2W(s)). \tag{3}$$

They are polynomials of degree $2n+1$, have zero $e^{-x}$ mean, and the completed trace has the decomposition

$$u_n = A_n^{\mathrm{tr}} + \sum_{m\ge 2} \frac{\Lambda(m)}{m^2} \Pi_n(\log m). \tag{4}$$

The zero-side form is

$$u_n = \sum_\rho \vartheta_\rho T_n(y_\rho), \qquad \vartheta_\rho = -\frac{c}{q(\rho)^2}, \qquad y_\rho = 1 - 2\vartheta_\rho. \tag{5}$$

The main theorem of [9] gives

$$\mathrm{RH} \iff \sup_n |u_n| < \infty \iff \limsup_{n\to\infty} |u_n|^{1/n} \le 1. \tag{6}$$

If RH fails, an escaping transported node creates an uncancellable pole in the trace generating function and hence a positive exponential Green rate.

## 3. The Krein block and the no-gain formulation

The elementary Chebyshev identities

$$T_m(y)T_n(y) + (1-y^2)U_{m-1}(y)U_{n-1}(y) = T_{|m-n|}(y), \tag{7}$$

$$T_m(y)T_n(y) - (1-y^2)U_{m-1}(y)U_{n-1}(y) = T_{m+n}(y) \tag{8}$$

are polynomial identities and therefore hold for complex $y$.

For $0 \le i, j \le N$, define

$$\mathcal{A}_N(i,j) = \sum_\rho \vartheta_\rho T_{N+i}(y_\rho) T_{N+j}(y_\rho), \tag{9}$$

$$\mathcal{U}_N(i,j) = \sum_\rho \vartheta_\rho(1-y_\rho^2)U_{N+i-1}(y_\rho)U_{N+j-1}(y_\rho). \tag{10}$$

Then (7)–(8) give

$$\mathsf{L}_N := \mathcal{A}_N + \mathcal{U}_N = [u_{|i-j|}]_{i,j=0}^N, \qquad \mathsf{H}_N := \mathcal{A}_N - \mathcal{U}_N = [u_{2N+i+j}]_{i,j=0}^N. \tag{11}$$

Thus

$$\mathcal{E}_N := \|\mathsf{L}_N\|_F^2 - \|\mathsf{H}_N\|_F^2 = 4\,\mathrm{tr}(\mathcal{A}_N\mathcal{U}_N). \tag{12}$$

On RH, $\mathcal{A}_N$ and $\mathcal{U}_N$ are positive semidefinite, so

$$-\mathsf{L}_N \preceq \mathsf{H}_N \preceq \mathsf{L}_N. \tag{13}$$

Conversely, the Wiener mean-square law in [9] implies that an off-line zero forces the high block to have a strictly larger exponential rate. In particular,

$$\text{RH} \iff \|\mathsf{H}_N\|_F \le \|\mathsf{L}_N\|_F \quad (N \ge 0). \tag{14}$$

A much weaker subexponential future-gain bound would also suffice: any estimate of the form

$$\|\mathsf{H}_N\|_F^2 \le N^{O(1)}\big(1 + \|\mathsf{L}_N\|_F^2\big) \tag{15}$$

forces subexponential trace energy and hence rules out the off-line Wiener exponent.

The purpose of the next sections is not to reprove (14), but to identify which pieces of (15) can be established unconditionally.

## 4. An $S_3$ current compression

Let $s_j = w^j(s)$ and

$$a_j = s_j^2 - s_j + 1, \qquad \ell_j = \ell(s_j).$$

Choose

$$\omega = e^{-2\pi i/3}$$

and define the discrete Fourier currents

$$C_k = \sum_{j=0}^{2} \omega^{-kj}\ell_j, \qquad \Gamma_k = \sum_{j=0}^{2}\omega^{-kj}a_j\ell_j, \qquad k \in \mathbb{Z}/3\mathbb{Z}. \tag{16}$$

Let

$$\alpha = e^{\pi i/3}, \qquad r = r(s) = \frac{s-\alpha}{s-\bar\alpha}.$$

Then $r(w(s)) = \omega r(s)$.

**Lemma 4.1** (Orbit Laplacian identity)**.** *One has*

$$s^2 - s + 1 = -\frac{3r}{(1-r)^2} \tag{17}$$

*and therefore*

$$(s^2 - s + 1)(2 - r - r^{-1}) = 3. \tag{18}$$

*Consequently the Fourier currents satisfy*

$$3C_k = 2\Gamma_k - r\Gamma_{k-1} - r^{-1}\Gamma_{k+1}. \tag{19}$$

*Proof.* Solving the cross-ratio relation for $s$ and simplifying gives (17); multiplying by $2 - r - r^{-1} = -(r-1)^2/r$ gives (18). Apply the same identity at the three orbit points, multiply by $\ell_j$, and take the discrete Fourier transform. Since $r(s_j) = \omega^j r$, the two shifted terms become $r\Gamma_{k-1}$ and $r^{-1}\Gamma_{k+1}$. □

Let

$$B_k = \sum_{j=0}^{2} \omega^{-kj} a_j, \qquad \beta_1 = \frac{B_1}{B_0}, \qquad \beta_2 = \frac{B_2}{B_0}.$$

A direct calculation gives

$$B_0 = q^2 + c, \qquad \beta_1 = \frac{2r^3+1}{3r^2}, \qquad \beta_2 = \frac{r^3+2}{3r}. \tag{20}$$

**Theorem 4.2** (Exact determinant cancellation)**.** *Put*

$$d = 1 - \beta_1\beta_2.$$

*Then*

$$d = \frac{6}{q^2+c}, \qquad B_0 d = 6, \tag{21}$$

*and*

$$\beta_2^2 - \beta_1 = -\frac{r}{2}d, \qquad \beta_1^2 - \beta_2 = -\frac{1}{2r}d. \tag{22}$$

*The standard currents therefore satisfy the denominator-free identities*

$$C_1 = \frac{1}{2}\big(\Gamma_1 - \beta_2\Gamma_2 - rC_0\big), \tag{23}$$

$$C_2 = \frac{1}{2}\big(\Gamma_2 - \beta_1\Gamma_1 - r^{-1}C_0\big), \tag{24}$$

*and the zero-mode current reduces to*

$$2\Gamma_0 = r^{-1}\Gamma_1 + r\Gamma_2 + 3C_0. \tag{25}$$

*Proof.* Substitute (20) and simplify. The identities (23)–(24) can also be obtained from the Fourier convolution

$$3\Gamma_k = \sum_{\nu=0}^{2} B_\nu C_{k-\nu}$$

by eliminating the other standard channel. The factor that would normally be $d^{-1}$ is canceled by $B_0 d = 6$. Equation (25) is the case $k = 0$ of (19). □

*Remark* 4.3 (The 8/9 identity). If $Z$ is the half-plane coordinate defined by

$$q^2 = c(Z^2 - 1),$$

then (21) becomes

$$d = \frac{8}{9Z^2}.$$

Thus the apparent near-null standard channel has an inverse-square defect. Theorem 4.2 shows that this defect is not an uncontrolled singularity in the native completed current.

The determinant is even more useful when combined with the trace weight.

**Corollary 4.4** (The determinant is the complementary Krein port)**.** *With* $W = c/q^2$ *and* $y = 1 + 2W$,

$$d = \frac{8}{9}\frac{W}{1+W}, \qquad \frac{W}{d} = \frac{9}{8}(1+W), \tag{26}$$

*and*

$$\boxed{\frac{W^2}{d} = -\frac{9}{32}(1-y^2).} \tag{27}$$

*Proof.* Use $q^2 = c/W$ in (21). Since

$$1 - y^2 = 1 - (1 + 2W)^2 = -4W(1 + W),$$

(27) follows. $\square$

Equation (27) is the exact bridge between the $S_3$ standard-channel Green factor and the $U$-polynomial block in (10). In particular, the normalized inverse determinant does not create a new quadratic weight; it converts the trace weight to the complementary Krein port.

## 5. An unconditional source contraction on the real invariant axis

We now use a zeta-specific fact that is nevertheless unconditional: $\log \xi(s)$ is strictly convex on the real axis. This follows, for example, from the classical positive even Laplace representation of $\xi(1/2 + u)$; the second derivative of the logarithm is the variance of the tilted measure. See [15] for the standard integral representation.

**Lemma 5.1** (Sign structure of the real cubic fiber)**.** *Let $q \in \mathbb{R} \setminus \{0\}$ and let $s_0, s_1, s_2$ be the corresponding cubic orbit. Then*

$$e_2 := \ell_0\ell_1 + \ell_1\ell_2 + \ell_2\ell_0 < 0. \tag{28}$$

*Proof.* It is enough to take the real representative $x > 2$. Its orbit is

$$x, \qquad \frac{x-1}{x}, \qquad -\frac{1}{x-1}.$$

Put

$$L = \ell(x), \qquad P = \ell\left(\frac{x-1}{x}\right), \qquad M = \ell\left(\frac{x}{x-1}\right).$$

The functional equation $\ell(1 - s) = -\ell(s)$ gives

$$\ell\left(-\frac{1}{x-1}\right) = -M.$$

Strict convexity and symmetry about $1/2$ imply

$$L > M > P > 0.$$

Hence

$$e_2 = LP - LM - PM = -L(M - P) - PM < 0.$$

The case $q < 0$ follows by reflection. $\square$

Define

$$\Sigma_+ = C_0^2 + 2C_1C_2, \qquad \Sigma_- = C_0^2 - 2C_1C_2. \tag{29}$$

For real $q$, $C_2 = \overline{C_1}$ and

$$C_1C_2 = \frac{1}{2}\left[(\ell_0 - \ell_1)^2 + (\ell_1 - \ell_2)^2 + (\ell_2 - \ell_0)^2\right] \geq 0. \tag{30}$$

Moreover

$$\Sigma_+ = 3\sum_{j=0}^{2} \ell_j^2, \qquad \Sigma_- = -\sum_{j=0}^{2} \ell_j^2 + 4e_2. \tag{31}$$

Thus Lemma 5.1 gives

$$-\Sigma_+ \leq \Sigma_- < -\frac{1}{3}\Sigma_+. \tag{32}$$

**Theorem 5.2** (Rank-one real-axis source contraction). *Let $q \in \mathbb{R} \setminus \{0\}$, so $W > 0$ and $y = 1 + 2W > 1$. For integers $m, n \geq 1$, define*

$$\begin{aligned} \mathfrak{F}_{mn}(q) &= C_0(q)^2 T_m(y) T_n(y) \\ &\quad + 2C_1(q)C_2(q)(1 - y^2)U_{m-1}(y)U_{n-1}(y). \end{aligned} \tag{33}$$

*Then*

$$\mathfrak{F}_{mn} = \frac{\Sigma_+}{2} T_{|m-n|}(y) + \frac{\Sigma_-}{2} T_{m+n}(y), \tag{34}$$

*and*

$$\mathfrak{F}_{mn} \leq C_0(q)^2 T_{|m-n|}(y). \tag{35}$$

*More generally, for any finite index set $I$,*

$$[\mathfrak{F}_{mn}]_{m,n\in I} - C_0(q)^2 [T_{|m-n|}(y)]_{m,n\in I} = \Sigma_- \, vv^T \preceq 0, \tag{36}$$

*where*

$$v_m = \sqrt{y^2 - 1}\, U_{m-1}(y).$$

*Thus the future component is a negative rank-one correction.*

*Proof.* Equations (7)–(8) give (34). Rewriting the same identity as

$$\mathfrak{F}_{mn} = C_0^2 T_{|m-n|} + \Sigma_-(y^2 - 1)U_{m-1}U_{n-1}$$

and using $\Sigma_- < 0$ proves both (35) and (36). $\square$

*Remark* 5.3 (Why this is not RH). The contraction in Theorem 5.2 lives on the real $q$ axis. The critical Hardy boundary introduced below corresponds to $\operatorname{Re} s = 1/2$, where $q$ is purely imaginary. Residue projection is not positivity preserving; indeed, the earlier cubic prime filters furnish explicit projected-square counterexamples [8]. Therefore Theorem 5.2 cannot simply be pushed through the residue as a Loewner inequality.

A further parity cancellation will be useful conceptually. Since reflection sends $q \mapsto -q$ and $\ell(1 - s) = -\ell(s)$, one has

$$C_0(q) = q\, \mathfrak{h}(q^2) \tag{37}$$

with $\mathfrak{h}$ meromorphic and regular at $q = 0$. Combining (37) with (27) gives

$$q^2 \frac{W^2}{d} = \frac{243}{32}(1 + W). \tag{38}$$

Thus the forced $q^2$ zero removes one full power of the quadratic Green weight.

## 6. The critical Cayley coordinate

We now turn to the local residue itself. The correct coordinate for the critical $e^{-3x}$ norm is not the Cayley coordinate centered at the Euler-product line used in [8], but the following one.

**Definition 6.1** (Critical Cayley coordinate). Set

$$\chi = \frac{s-2}{s+1}, \qquad s = \frac{2+\chi}{1-\chi}. \tag{39}$$

**Proposition 6.2** (Critical geometry). *The coordinate* (39) *has the properties*

$$|\chi| = 1 \iff \operatorname{Re} s = \frac{1}{2}, \tag{40}$$

$$s \mapsto 1 - s \quad \iff \quad \chi \mapsto \chi^{-1}, \tag{41}$$

*and*

$$2 \mapsto 0, \qquad \frac{1}{2} \mapsto -1, \qquad -1 \mapsto \infty.$$

*Moreover*

$$q\left(\frac{2+\chi}{1-\chi}\right) = -\frac{27\chi(\chi+1)}{2(\chi-1)(\chi+2)(2\chi+1)}, \tag{42}$$

*and*

$$W(\chi) = \frac{(\chi-1)^2(\chi+2)^2(2\chi+1)^2}{27\chi^2(\chi+1)^2}. \tag{43}$$

*Proof.* All statements follow by direct substitution. In particular,

$$|s-2| = |s+1|$$

is the perpendicular bisector of the segment $[-1,2]$, namely $\operatorname{Re} s = 1/2$. □

This coordinate also makes the weight $e^{-3x}$ canonical. The ordinary Laguerre generating function gives

$$\frac{1}{1-\chi}\exp\left(-\frac{3x\chi}{1-\chi}\right) = \sum_{k\geq 0} L_k(3x)\chi^k. \tag{44}$$

Since $s-2 = 3\chi/(1-\chi)$,

$$e^{-x(s-2)} = \frac{3}{s+1}\sum_{k\geq 0} L_k(3x)\left(\frac{s-2}{s+1}\right)^k. \tag{45}$$

Thus the critical line, the local residue, and the $e^{-3x}$ orthogonality all occur in the same disk coordinate.

## 7. The cubic Schur multiplier

Let

$$y(\chi) = 1 + 2W(\chi).$$

Choose the local Joukowski branch $\lambda(\chi)$ near $\chi = 0$ by

$$y = \frac{1}{2}(\lambda + \lambda^{-1}), \qquad \lambda(0) = 0. \tag{46}$$

Then

$$W = \frac{(1-\lambda)^2}{4\lambda}. \tag{47}$$

Let

$$\psi(\chi) = \frac{2(\chi^2+\chi+1)^{3/2} - (\chi-1)(\chi+2)(2\chi+1)}{3\sqrt{3}\,(\chi+1)}, \tag{48}$$

where the square root is the analytic branch in $\mathbb{D}$ normalized by

$$(\chi^2+\chi+1)^{1/2} = 1 + O(\chi)$$

at the origin. The zeros of $\chi^2+\chi+1$ lie on $\partial\mathbb{D}$, so this branch is well defined in $\mathbb{D}$. Put

$$\phi(\chi) = \psi(\chi)^2. \tag{49}$$

**Proposition 7.1** (Joukowski factorization)**.** *The branch* (46) *satisfies*

$$\boxed{\lambda(\chi) = \frac{\chi^2}{\phi(\chi)}.} \tag{50}$$

*In particular,*

$$\phi(0) = \frac{16}{27}. \tag{51}$$

*Proof.* Insert (43) into the quadratic relation

$$\lambda + \lambda^{-1} = 2 + 4W$$

and solve for the branch satisfying $\lambda(0) = 0$. Simplifying the resulting square root gives (48)–(50). At $\chi = 0$,

$$\psi(0) = \frac{4}{3\sqrt{3}},$$

which gives (51). □

**Lemma 7.2** (Boundary regularity)**.** *The apparent singularity of $\psi$ at $\chi = -1$ is removable, and*

$$\psi(\chi) = \frac{3\sqrt{3}}{4}(\chi + 1) + O((\chi + 1)^2) \qquad (\chi \to -1). \tag{52}$$

*In particular $\psi(-1) = 0$. At $\chi = 1$,*

$$\phi(1) = 1, \qquad \phi'(1) = 2 - \sqrt{3}. \tag{53}$$

*The algebraic branch in* (48) *therefore extends continuously to $\overline{\mathbb{D}}$; at the two roots of $\chi^2 + \chi + 1$ the first numerator term simply tends to zero.*

*Proof.* At $\chi = -1$ both numerator and denominator in (48) vanish. Taylor expansion of the analytic branch from $\mathbb{D}$ gives (52). At $\chi = 1$ the square-root branch is regular and positive, and direct differentiation gives

$$\psi(1) = 1, \qquad \psi'(1) = 1 - \frac{\sqrt{3}}{2}.$$

Since $\phi = \psi^2$, (53) follows. □

The essential point is not merely that $16/27 < 1$, but that the entire multiplier is contractive. We use standard Hardy-space notation as in [3].

**Theorem 7.3** (Critical Schur theorem)**.** *The function $\psi$ is Schur:*

$$|\psi(\chi)| \le 1 \qquad (\chi \in \mathbb{D}). \tag{54}$$

*Consequently*

$$\|\phi\|_{H^\infty(\mathbb{D})} \le 1. \tag{55}$$

*Proof.* It is enough to verify the radial boundary inequality. Let $\chi = e^{i\theta}$ and put

$$c_\theta = 1 + 2\cos\theta \in [-1, 3].$$

Then

$$\chi^2 + \chi + 1 = \chi c_\theta,$$

and

$$|(\chi - 1)(\chi + 2)(2\chi + 1)|^2 = (3 - c_\theta)(3 + 2c_\theta)^2. \tag{56}$$

Moreover, after division by $\chi^{3/2}$, the latter cubic factor is purely imaginary, since

$$\frac{[(\chi-1)(\chi+2)(2\chi+1)]^2}{\chi^3} = (c_\theta - 3)(2c_\theta + 3)^2.$$

If $c_\theta \geq 0$, the two terms in the numerator of (48) are orthogonal in phase. Hence

$$\begin{aligned}|\text{numerator}|^2 &= 4c_\theta^3 + (3 - c_\theta)(3 + 2c_\theta)^2 \\ &= 27(c_\theta + 1) = |3\sqrt{3}(\chi + 1)|^2.\end{aligned}$$

Thus $|\psi| = 1$ on this arc.

If $c_\theta = -d$ with $0 \leq d < 1$, the two terms are collinear and the boundary inequality reduces to

$$A_d := (3 - 2d)\sqrt{d+3} - 2d^{3/2} \geq 0. \tag{57}$$

Both sides in

$$(3 - 2d)\sqrt{d+3} \geq 2d^{3/2}$$

are nonnegative, and squaring gives

$$(3 - 2d)^2(d + 3) - 4d^3 = 27(1 - d) \geq 0,$$

so (57) holds. A direct expansion then gives

$$27(1 - d) - A_d^2 = 4d^{3/2}A_d \geq 0,$$

which is exactly $|\psi(e^{i\theta})| \leq 1$ on the remaining open arc. At the endpoint $d = 1$, namely $\chi = -1$, Lemma 7.2 gives $\psi(-1) = 0$. Thus $|\psi| \leq 1$ on the entire boundary. Since the same lemma gives a continuous extension to $\overline{\mathbb{D}}$, the maximum modulus principle proves (54). Equation (55) follows from $\phi = \psi^2$. □

## 8. Critical Laguerre–Hardy representation of the trace filters

Set

$$F_n(s) = W(s)T_n(1 + 2W(s)). \tag{58}$$

By (45), the residue definition (3) gives an exact critical Laguerre expansion.

**Proposition 8.1** (Critical Laguerre coefficients)**.** *For every $n \geq 0$,*

$$\Pi_n(x) = \sum_{k=0}^{2n+1} b_{n,k}L_k(3x), \tag{59}$$

*where*

$$b_{n,k} = -\operatorname*{Res}_{s=2} F_n(s)\frac{3}{s+1}\left(\frac{s-2}{s+1}\right)^k. \tag{60}$$

*Consequently*

$$I_n^{(0)} := \int_0^\infty e^{-3x}|\Pi_n(x)|^2\,dx = \frac{1}{3}\sum_{k=0}^{2n+1}|b_{n,k}|^2. \tag{61}$$

*Proof.* Substitute (45) into (3). Near $s = 2$, $W$ has a double pole, while $T_n(1 + 2W)$ is a degree-$n$ polynomial in $W$. Hence $F_n = WT_n(1 + 2W)$ has pole order $2n + 2$, so the last possible Laguerre index is $2n + 1$. Finally,

$$\int_0^\infty e^{-3x}L_j(3x)L_k(3x)\,dx = \frac{1}{3}\delta_{jk},$$

which proves (61). □

For $n \geq 1$, the Joukowski representation removes the analytic Chebyshev branch. Indeed,

$$F_n = \frac{(1-\lambda)^2}{8}\big(\lambda^{n-1} + \lambda^{-n-1}\big). \tag{62}$$

The first term in (62) is analytic at $\chi = 0$ and contributes no residue. Thus only

$$F_n^{\mathrm{sing}} = \frac{1}{8}(1-\lambda)^2\lambda^{-n-1} \tag{63}$$

matters.

Define

$$G(\chi) = \frac{(\phi(\chi) - \chi^2)^2}{1-\chi}. \tag{64}$$

**Lemma 8.2** (The Hardy prefactor)**.** *The function $G$ extends holomorphically across $\chi = 1$ and belongs to $H^\infty(\mathbb{D})$. More precisely,*

$$\phi(\chi) - \chi^2 = -\sqrt{3}\,(\chi - 1) + O((\chi-1)^2), \tag{65}$$

*and hence*

$$G(\chi) = 3(1-\chi) + O((1-\chi)^2) \qquad (\chi \to 1). \tag{66}$$

*Proof.* By Lemma 7.2,

$$\phi'(1) - 2 = (2 - \sqrt{3}) - 2 = -\sqrt{3},$$

which gives (65) and (66). Thus the only possible singularity arising from the denominator in (64) is removable. On the complement of a neighborhood of $\chi = 1$, the denominator is bounded away from zero and $\phi$ is bounded by Theorem 7.3. Hence $G$ is bounded on $\mathbb{D}$. $\square$

**Proposition 8.3** (Isometric arc and sharpness)**.** *Let*

$$\mathscr{A} = \{e^{i\theta} : 1 + 2\cos\theta \geq 0\}.$$

*Then*

$$|\phi(e^{i\theta})| = 1 \quad \text{on } \mathscr{A},$$

*whereas $|\phi| < 1$ in the interior of the complementary boundary arc. Moreover $G$ has nonzero boundary mass on $\mathscr{A}$; for example*

$$|G(i)|^2 = \frac{5000}{729}.$$

*Consequently*

$$\lim_{m\to\infty} \|G\phi^m\|_{H^2}^2 = \frac{1}{2\pi}\int_{\mathscr{A}} |G(e^{i\theta})|^2\, d\theta > 0. \tag{67}$$

*Thus inner–outer factorization of $\phi$ cannot, by itself, improve the uniform Hardy contraction below to decay of the full $H^2$ norm.*

*Proof.* The boundary-modulus statements are exactly the two cases in the proof of Theorem 7.3; on the complementary open arc the inequality is strict because

$$27(1-d) - A_d^2 = 4d^{3/2}A_d > 0 \qquad (0 < d < 1).$$

Direct substitution at $\chi = i$ gives the displayed value of $|G(i)|^2$. Since $G \in H^\infty$ and $|\phi| \leq 1$, dominated convergence on the unit circle gives (67). $\square$

**Theorem 8.4** (Exact Hardy coefficient formula)**.** *For $n \geq 1$ and $0 \leq k \leq 2n+1$,*

$$\boxed{b_{n,k} = -\frac{3}{8}[\chi^{2n+1-k}]G(\chi)\phi(\chi)^{n-1}.} \tag{68}$$

*Equivalently, if*

$$B_n(\zeta) = \sum_{k=0}^{2n+1} b_{n,k}\zeta^k,$$

*then*

$$B_n(\zeta) = \frac{3}{1-\zeta}\widehat{\Pi}_n\left(\frac{3}{1-\zeta}\right), \tag{69}$$

*where* $\widehat{\Pi}_n(p) = \int_0^\infty e^{-px}\Pi_n(x)\,dx$.

*Proof.* Using (50) in (63),

$$F_n^{\mathrm{sing}} = \frac{1}{8}(\phi - \chi^2)^2\phi^{n-1}\chi^{-2n-2}.$$

Changing the residue from $s$ to $\chi$ gives

$$ds = \frac{3}{(1-\chi)^2}\,d\chi,$$

while the coefficient factor in (60) is $(1-\chi)\chi^k$. Taking the coefficient of $\chi^{-1}$ yields (68). Formula (69) is the ordinary Cayley change of variables in the Laplace transform; on $|\zeta| = 1$ one has $\mathrm{Re}(3/(1-\zeta)) = 3/2$. □

We can now prove the critical filter estimate that had previously appeared only numerically.

**Theorem 8.5** (Uniform critical Hardy contraction)**.** *There exists an absolute constant $C < \infty$ such that*

$$\boxed{\int_0^\infty e^{-3x}|\Pi_n(x)|^2\,dx \leq C \qquad (n \geq 0).} \tag{70}$$

*For $n \geq 1$ one may take*

$$C = \frac{3}{64}\,\|G\|_{H^2}^2 \tag{71}$$

*up to enlarging the constant to include $n = 0$.*

*Proof.* By (68), the vector $(b_{n,k})_{k=0}^{2n+1}$ is, up to reversal and the factor $3/8$, an initial coefficient segment of $G\phi^{n-1}$. Hence Parseval and Theorem 7.3 give

$$\begin{aligned} I_n^{(0)} &= \frac{1}{3}\sum_k |b_{n,k}|^2 \\ &\leq \frac{3}{64}\left\|G\phi^{n-1}\right\|_{H^2}^2 \\ &\leq \frac{3}{64}\,\|G\|_{H^2}^2\,. \end{aligned}$$

The case $n = 0$ is finite directly from $\Pi_0$. □

*Remark* 8.6 (Why the theorem is a uniform bound, not a decay theorem). Proposition 8.3 shows that the multiplier is genuinely isometric on a boundary arc of positive measure and that $G$ does not vanish there. Thus the simple Hardy estimate used in Theorem 8.5 is not hiding an automatic decay factor. Any sharper asymptotic for the truncated coefficient energy $I_n^{(0)}$ would require information beyond the inner–outer decomposition of $\phi$.

**Corollary 8.7** (Contractive top edge)**.** *For $n \geq 1$,*

$$\boxed{b_{n,2n+1} = -\frac{2}{9}\left(\frac{16}{27}\right)^n.} \tag{72}$$

*Proof.* Set $k = 2n+1$ in (68). Since

$$G(0) = \phi(0)^2,$$

we obtain

$$b_{n,2n+1} = -\frac{3}{8}\phi(0)^{n+1} = -\frac{3}{8}\left(\frac{16}{27}\right)^{n+1} = -\frac{2}{9}\left(\frac{16}{27}\right)^n.$$

□

Thus the geometric boundary multiplier that was singular before residue extraction becomes a strict local contraction after the critical Hardy projection.

## 9. Critical moment bounds

Define

$$I_n^{(j)} = \int_0^\infty x^j e^{-3x} |\Pi_n(x)|^2 \, dx, \qquad j = 0, 1, 2. \tag{73}$$

The higher critical moments require no new asymptotic theorem.

**Lemma 9.1** (Multiplication by $x$ in the critical Laguerre basis)**.** *For ordinary Laguerre polynomials,*

$$xL_k(3x) = \frac{1}{3}\big[(2k+1)L_k(3x) - (k+1)L_{k+1}(3x) - kL_{k-1}(3x)\big]. \tag{74}$$

*If $P$ has degree at most $d$, then*

$$\|xP\|_{L^2(e^{-3x}dx)} \leq \frac{4d+2}{3} \|P\|_{L^2(e^{-3x}dx)}. \tag{75}$$

*Proof.* Equation (74) is the standard three-term Laguerre recurrence after the scaling $y = 3x$; see, for example, [14]. In the orthogonal basis $L_k(3x)$ the multiplication matrix has maximum absolute column sum at most $(4d+2)/3$ and no larger row sum. The Schur matrix bound gives (75). □

**Corollary 9.2** (Critical first and second moments)**.** *For every $n \geq 0$,*

$$I_n^{(1)} = O(n), \tag{76}$$

*and*

$$\boxed{I_n^{(2)} = O(n^2).} \tag{77}$$

*More explicitly, for $d = 2n+1$,*

$$I_n^{(2)} \leq \left(\frac{4d+2}{3}\right)^2 I_n^{(0)}, \tag{78}$$

*and*

$$I_n^{(1)} \leq \frac{4d+2}{3} I_n^{(0)}. \tag{79}$$

*Proof.* The second-moment estimate is (75) applied to $P = \Pi_n$. The first-moment estimate follows from Cauchy–Schwarz:

$$I_n^{(1)} \leq (I_n^{(0)} I_n^{(2)})^{1/2}.$$

Now apply Theorem 8.5. □

## 10. A polynomial prime-square sampling theorem

The critical $e^{-3x}$ norm is exactly strong enough to pay for the spacing of the logarithmic prime-power samples without changing the exponential scale.

**Lemma 10.1** (Logarithmic sampling). *Let $P$ be a polynomial of degree at most $d$. Then*

$$\sum_{m\geq 2}\frac{(\log m)^2}{m^4}|P(\log m)|^2 \ll d^4\,\|P\|^2_{L^2(e^{-3x}dx)}\,, \tag{80}$$

*with an absolute implied constant.*

*Proof.* Let

$$x_m=\log m,\qquad I_m=[\log m,\log(m+1)],\qquad \delta_m=|I_m|.$$

The elementary one-dimensional Sobolev estimate gives

$$|P(x_m)|^2\leq\frac{2}{\delta_m}\int_{I_m}|P(x)|^2\,dx+2\delta_m\int_{I_m}|P'(x)|^2\,dx. \tag{81}$$

Since $\delta_m\asymp m^{-1}$ and $e^{-3x}\asymp m^{-3}$ on $I_m$, multiplication by $(\log m)^2m^{-4}$ and summation give

$$\sum_{m\geq 2}\frac{(\log m)^2}{m^4}|P(\log m)|^2\ll\|xP\|^2_{L^2(e^{-3x})}+\left\|xP'\right\|^2_{L^2(e^{-3x})}. \tag{82}$$

The derivative identity

$$\frac{d}{dx}L_k(3x)=-3\sum_{j=0}^{k-1}L_j(3x) \tag{83}$$

shows, by the same matrix norm estimate, that

$$\left\|P'\right\|_{L^2(e^{-3x})}\leq 3d\,\|P\|_{L^2(e^{-3x})}\,. \tag{84}$$

Apply Lemma 9.1 to $P$ and $P'$. The first term in (82) is $O(d^2)\,\|P\|^2$ and the second is $O(d^4)\,\|P\|^2$, proving (80). ☐

**Corollary 10.2** (Prime-square filter bound). *For the trace filters,*

$$\boxed{\sum_{m\geq 2}\frac{\Lambda(m)^2}{m^4}|\Pi_n(\log m)|^2=O(n^4).} \tag{85}$$

*Proof.* Use $\Lambda(m)\leq\log m$, Lemma 10.1, and Theorem 8.5. ☐

We record the implication for the block quadratic variation. Writing

$$\mathcal{E}_N=\sum_{n=0}^{4N}w_{N,n}|u_n|^2,$$

the weights are explicitly

$$w_{N,n}=\begin{cases}N+1, & n=0,\\ 2(N+1-n), & 1\leq n\leq N,\\ 0, & N<n<2N,\\ -(k+1), & n=2N+k,\ 0\leq k\leq N,\\ -(2N+1-k), & n=2N+k,\ N<k\leq 2N.\end{cases} \tag{86}$$

Thus

$$\sum_{n=0}^{4N} |w_{N,n}| = 2(N+1)^2, \qquad \sum_{n=0}^{4N} w_{N,n} = 0.$$

All active indices satisfy $n \le 4N$. Therefore the jump term

$$\mathcal{J}_N = \sum_{m\ge 2} \frac{\Lambda(m)^2}{m^4} \sum_n w_{N,n} \Pi_n(\log m)^2 \tag{87}$$

satisfies

$$\boxed{|\mathcal{J}_N| = O(N^6).} \tag{88}$$

The sign of $\mathcal{J}_N$ is not fixed; the content of (88) is that it cannot carry the exponential future gain that an off-line zero would require.

## 11. The continuous transport form

For completeness we state the exact jump-removal identity that places (88) in context. Put

$$\Psi(X) = \sum_{m\le X} \Lambda(m), \qquad R(X) = \Psi(X) - X, \qquad f_n(X) = X^{-2}\Pi_n(\log X),$$

and define the truncated completed current

$$U_n(X) = A_n^{\mathrm{tr}} + \int_{1^-}^{X} f_n(t)\, dR(t).$$

Then

$$dU_n = f_n\, dR.$$

Define

$$V_n(X) = U_n(X) - R(X) f_n(X). \tag{89}$$

Since $f_n$ is continuous,

$$\boxed{dV_n(X) = -R(X) f_n'(X)\, dX.} \tag{90}$$

Thus all prime-power jumps cancel exactly. The classical Chebyshev estimate $\Psi(X) = O(X)$ gives $R(X) = O(X)$, while

$$f_n(X) = O\big(X^{-2}(\log X)^{2n+1}\big)$$

for fixed $n$. Therefore $R(X) f_n(X) \to 0$ unconditionally, and

$$u_n = B_n - \int_1^\infty R(X) f_n'(X)\, dX, \tag{91}$$

where $B_n = V_n(1^-)$. For the signed block weights,

$$\mathcal{E}_N = \mathcal{E}_N^{(B)} - 2\int_1^\infty R(X)\mathcal{B}_N(X)\, dX + \iint R(X)R(Y)\mathcal{C}_N(X,Y)\, dX\, dY, \tag{92}$$

with

$$\mathcal{B}_N(X) = \sum_n w_{N,n} B_n f_n'(X), \qquad \mathcal{C}_N(X,Y) = \sum_n w_{N,n} f_n'(X) f_n'(Y).$$

Equation (88) shows that the discrete quadratic variation is polynomial. The remaining issue is the polarized continuous arithmetic source, not the local filter norm.

## 12. The completed source in the critical Hardy coordinate

The critical Hardy factorization also gives a precise formula for the completed endpoint residue. Define

$$\mathcal{H}(\chi) = \frac{1}{1-\chi}\ell\left(\frac{2+\chi}{1-\chi}\right) = \sum_{j\geq 0} h_j \chi^j \tag{93}$$

whenever the Taylor series is valid at the origin.

**Theorem 12.1** (Exact completed endpoint coefficient identity)**.** *For $n \geq 1$,*

$$\operatorname*{Res}_{s=2} \ell(s)F_n(s) = -\sum_{k=0}^{2n+1} h_k b_{n,k}, \tag{94}$$

*and equivalently*

$$\boxed{\operatorname*{Res}_{s=2} \ell(s)F_n(s) = \frac{3}{8}[\chi^{2n+1}]\mathcal{H}(\chi)G(\chi)\phi(\chi)^{n-1}.} \tag{95}$$

*Proof.* Expand $\mathcal{H}$ at the origin and use (60); this gives (94). Substitute (68) and collect the convolution coefficient to obtain (95). □

The formula (95) sharply separates the filter from the arithmetic source. The factor $G\phi^{n-1}$ is uniformly controlled in $H^2$ by Theorem 8.5. The only uncontrolled analytic object is $\mathcal{H}$.

**Proposition 12.2** (The remaining pole is exactly the RH pole)**.** *The function $\mathcal{H}$ is holomorphic in $\mathbb{D}$ if and only if $\xi$ has no zero in the open half-plane* $\operatorname{Re} s > 1/2$. *By the functional equation, this is equivalent to RH.*

*Proof.* The Cayley map (39) sends $\mathbb{D}$ biholomorphically to $\operatorname{Re} s > 1/2$. The logarithmic derivative $\ell = \xi'/\xi$ is meromorphic, with poles exactly at the zeros of $\xi$, counted with multiplicity. The prefactor $(1-\chi)^{-1}$ introduces no pole in $\mathbb{D}$. Thus interior poles of $\mathcal{H}$ are exactly the images of zeros with $\operatorname{Re} \rho > 1/2$. The functional equation reflects any zero with $\operatorname{Re} \rho < 1/2$ to one with $\operatorname{Re} \rho > 1/2$. □

*Remark* 12.3 (The central completion). The full trace residue also contains the explicit central contribution at $s = 1/2$, which corresponds to the boundary point $\chi = -1$. Theorem 12.1 is the exact endpoint identity at $s = 2$; it does not discard the central completion. The point is that the interior pole set of the endpoint source is already exactly the off-line zero set in the right half-plane.

## 13. Circularity audit and a no-go model

Theorem 12.1 makes a tempting but invalid argument visible. If one assumed

$$\mathcal{H} \in H^2(\mathbb{D}),$$

then Cauchy–Schwarz and the uniform bound for $G\phi^{n-1}$ would immediately control the endpoint trace coefficients. But Proposition 12.2 shows that holomorphy of $\mathcal{H}$ in the disk already excludes every off-line zero. Such an $H^2$ assumption would therefore be circular.

Nor can the real-axis source contraction alone exclude the poles. The following elementary model shows why generic symmetry, positivity, and log-convexity are insufficient.

**Proposition 13.1** (A symmetric log-convex countermodel)**.** *The entire function*

$$M(s) = \cosh\left(s - \frac{1}{2}\right) + \frac{1}{2}\cosh(2s-1) \tag{96}$$

*has the following properties:*

(i) $M(1-s) = M(s)$;
(ii) $M(s) > 0$ *for real* $s$;
(iii) $\log M(s)$ *is strictly convex on the real axis;*
(iv) $M$ *nevertheless has zeros off the symmetry line* $\operatorname{Re} s = 1/2$.

*Proof.* The first two statements are immediate. On the real axis $M$ is a positive nondegenerate sum of exponentials, so the logarithm is strictly convex by the standard log-sum-exp variance identity. Put $u = s - 1/2$ and $c = \cosh u$. Since $\cosh 2u = 2c^2 - 1$,

$$M = 0 \iff c^2 + c - \frac{1}{2} = 0.$$

One root is

$$c = -\frac{1+\sqrt{3}}{2} < -1.$$

Hence

$$u = \pm \operatorname{arcosh}\left(\frac{1+\sqrt{3}}{2}\right) + i\pi$$

modulo the usual $2\pi i$ periods, and these zeros have nonzero real part. □

Thus the real-axis contraction in Theorem 5.2 is meaningful but cannot, by itself, be promoted to a proof of RH by a generic maximum principle. Some genuinely zeta-specific arithmetic coupling must still exclude the poles in Proposition 12.2.

## 14. What has been contracted, and what remains

The cumulative effect of the preceding theorems can be stated cleanly.

**Theorem 14.1** (Reduction after critical contraction)**.** *For the cubic Chebyshev trace, the following statements are unconditional.*

(a) *The normalized* $S_3$ *standard-channel determinant singularity cancels exactly in the native current; its quadratic Green factor is the Krein* $U$*-port* (27).
(b) *On the real invariant axis, the completed standard source has a rank-one negative future component* (36).
(c) *The exact trace prime filters satisfy the uniform critical Hardy bound* (70).
(d) *Their weighted critical moments satisfy* $I_n^{(1)} = O(n)$ *and* $I_n^{(2)} = O(n^2)$.
(e) *Their prime-square sampling energy is* $O(n^4)$*, and the signed block quadratic variation is* $O(N^6)$.
(f) *The endpoint completed trace coefficient has the exact form* (95)*; in that formula the filter factor is uniformly Hardy-contractile, while every possible interior exponential instability is carried by a pole of* $\mathcal{H}$*, equivalently by a zero of* $\xi$ *with* $\operatorname{Re}\rho > 1/2$.

*Proof.* This is a collection of Theorems 4.2, 5.2, 8.5, Corollary 9.2, Corollary 10.2, and Theorem 12.1 together with Proposition 12.2. □

Theorem 14.1 is the strongest conclusion justified by the present argument. It is more than a new equivalence: the local filter hierarchy and its discrete quadratic variation are now bounded independently of RH. At the same time it is less than a proof: the remaining pole content is the arithmetic zero-location problem itself.

A complete proof through this architecture would require a zeta-specific theorem controlling the pole-sensitive bilinear coupling in the continuous completed flux without assuming that $\mathcal{H}$ is holomorphic in $\mathbb{D}$. The present paper does not supply such a theorem. Its contribution is to show that no exponential obstruction remains in the local Chebyshev filter, the critical residue geometry, the standard-channel determinant, or the prime-square jump term. Any exponential future gain must enter through the completed arithmetic source.

This distinction is important. The earlier boundary-layer calculation [10] showed that residue-first extraction factorially regularizes a local geometric singularity. The critical Schur theorem upgrades that local observation to a global uniform $H^2$ contraction for the exact trace filters. The $S_3$ identities show, independently, that the apparent two-channel inverse defect is the same complementary port already present in the Krein block. The two mechanisms therefore agree on where the geometry is benign. What they do not do is erase the poles of $\xi'/\xi$.

## References


[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver & Boyd, Edinburgh, 1965.

[2] E. Bombieri and J. C. Lagarias, Complements to Li's criterion for the Riemann hypothesis, *J. Number Theory* **77** (1999), 274–287.

[3] P. L. Duren, *Theory of $H^p$ Spaces*, Pure and Applied Mathematics, vol. 38, Academic Press, New York, 1970.

[4] M. Griffin, K. Ono, L. Rolen, and D. Zagier, Jensen polynomials for the Riemann zeta function and other sequences, *Proc. Natl. Acad. Sci. USA* **116** (2019), 11103–11110.

[5] X.-J. Li, The positivity of a sequence of numbers and the Riemann hypothesis, *J. Number Theory* **65** (1997), 325–333.

[6] Y. V. Matiyasevich, Hidden life of Riemann's zeta function. 2. Electrons and trains, preprint, 2007, arXiv:0709.0028.

[7] C. O'Sullivan, Zeros of Jensen polynomials and asymptotics for the Riemann xi function, *Res. Math. Sci.* **8** (2021), Article 46.

[8] B. Rouyea and C. Bourgeois, *The Cubic Descent of the Riemann Xi Function: A One-Tower Hankel Criterion for the Riemann Hypothesis*, preprint, 2026, alphaXiv: `https://www.alphaxiv.org/abs/2608.riemann-xi-hankel-criterion`.

[9] B. Rouyea and C. Bourgeois, *A Chebyshev Trace Criterion for the Riemann Hypothesis*, preprint, 2026, alphaXiv: `https://www.alphaxiv.org/abs/2608.chebyshev-trace-riemann-hypothesis`.

[10] B. Rouyea and C. Bourgeois, *A Boundary-Layer Calculus for Chebyshev Prime Filters*, companion manuscript, 2026.

[11] M. Suzuki, Aspects of the screw function corresponding to the Riemann zeta function, preprint, 2022, arXiv:2206.03682.

[12] J. Sondow and C. Dumitrescu, A monotonicity property of Riemann's xi function and a reformulation of the Riemann Hypothesis, preprint, 2010, arXiv:1005.1104.

[13] M. Suzuki, A family of deformations of the Riemann xi-function, *Acta Arith.* **157** (2013), 201–238.

[14] G. Szegő, *Orthogonal Polynomials*, 4th ed., American Mathematical Society, Providence, RI, 1975.

[15] E. C. Titchmarsh, revised by D. R. Heath-Brown, *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford University Press, Oxford, 1986.