Real and Complex Analysis Notes

1. Sequences and Limits

> "No one shall expel us from the paradise that Cantor has created for us."
>
> *David Hilbert*

1.1 Structure of these notes

This chapter works through early analysis exercises using clear algebra and first principles. It solves inequalities and presents solution sets on number lines. It develops supremum and infimum arguments for sets built from bounded sets. It proves sequence convergence using epsilon definitions and uses monotone convergence to find a limit. It ends with epsilon delta limit proofs, including limits that do not exist and limits at infinity.

1.2 Exercises

Exercise 1.1 Solve the following inequalities. Show all your working and simplify the result.

1.
$$7 - x \leq 2 + 3x$$

2.
$$\frac{x-6}{1-2x} \leq 0$$

3.
$$|2x-3| < 3$$

For the first one proceed as follows:

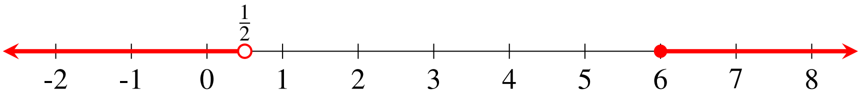
$$\begin{aligned} 7 - x &\leq 2 + 3x \\ 5 &\leq 4x \\ x &\geq \frac{5}{4} \end{aligned}$$

which is, in set notation, the solution $\{x : x \geq \frac{5}{4}\}$. For the second one there are two cases, either

$$\{x - 6 \geq 0 \text{ and } 1 - 2x < 0\} \quad \text{or} \quad \{x - 6 \leq 0 \text{ and } 1 - 2x > 0\}$$

the first case implies that $x > \frac{1}{2}$ and $x \geq 6$, which means we require $x \geq 6$. The second case implies $x \leq 6$ and $x < \frac{1}{2}$, which means we require $x < \frac{1}{2}$. This gives our solution:

$$\{x : x < \tfrac{1}{2}\} \cup \{x : x \geq 6\}$$



For this last one simply see:

$$\begin{aligned} |2x-3| &< 3 \\ -3 < 2x-3 &< 3 \\ 0 < 2x &< 6 \\ 0 < x &< 3 \end{aligned}$$

so our solution is

$$\{x : 0 < x < 3\}$$

Exercise 1.2 For a bounded nonempty set S, we define $\sup S$ to be the least upper bound of set S, and we define $\inf S$ to be the greatest lower bound of S. Using the definition, show that

1. D is nonempty.
2. D is bounded (above and below).
3. $\sup D = \sup A - 2\inf B$.

Here we have used the following definition of D:

Definition 1.2.1 Let A and B be two *non-empty and bounded* subsets of $\mathbb{R}$, then we define:

$$D := \{a - 2b : a \in A, b \in B\}$$

We begin by showing that D is nonempty (first proof). We then establish an upperbound for D (second proof). Since the lowerbound can be found in virtually the same way, it is given in the remark at the end. Together, these show that D is bounded. The last proof finds the supremum of D.

Proof. Since A and B are non-empty, we know there exist at least one element $a_0 \in A$ and $b_0 \in B$. Then D has at least one element $a_0 - 2b_0 \in D$. This shows that D is non-empty. ■

Proof. Since both A and B are non-empty and bounded, they each have a supremum and infimum. Set $\alpha_A = \sup A$. Then $\alpha_A \geq a$ for all $a \in A$. Let $\beta_B = \inf B$. Then $\beta_B \leq b$ for all $b \in B$. This is the same as $-2\beta_B \geq -2b$ for all $b \in B$. We then add these like so:

$$\begin{aligned} \alpha_A &\geq a && \forall a \in A \\ -2\beta_B &\geq -2b && \forall b \in B \\ \alpha_A - 2\beta_B &\geq a - 2b && \forall a \in A, \forall b \in B \\ \alpha_A - 2\beta_B &\geq d && \forall d \in D \end{aligned}$$

thus $\alpha_A - 2\beta_B$ is an upperbound for D. ■

Because we know that D is bounded above, we know that it has a supremum. We will prove that this is exactly $\sup A - 2\inf B$.

Proof. We will show that $\sup D = \alpha_A - 2\beta_B$. For this to hold, we require two things. First, that $\alpha_A - 2\beta_B$ is an upperbound for D. This was already proven. Secondly, we need to show that for any upperbound M of D then

$$M \geq \alpha_A - 2\beta_B$$

Suppose M is an upper bound of D, then

$$\begin{aligned} M &\geq d && \forall d \in D \\ M &\geq a - 2b && \forall a \in A,\ \forall b \in B \\ M + 2b &\geq a && \forall a \in A,\ \forall b \in B \end{aligned}$$

this last line implies that for every $b \in B$, $M + 2b$ is an upperbound for A. Therefore, $M + 2b \geq \alpha_A$ by definition of α_A. Thus, $M + 2b \geq \alpha_A$ for all $b \in B$. Rearranging we can get:

$$\begin{aligned} M &\geq \alpha_A - 2b && \forall b \in B \\ -\frac{M - \alpha_A}{2} &\leq b && \forall b \in B \end{aligned}$$

this last equation establishes a lowerbound for B, which implies it is less than or equal to the infimum of B. Thus:

$$\begin{aligned} -\frac{M - \alpha_A}{2} &\leq \beta_B \\ M &\geq \alpha_A - 2\beta_B \end{aligned}$$

This shows that for any upperbound of D, that $\sup A - 2\inf B$ is less than or equal to that upperbound, completing the proof. ■

R This remark completes the proof that D is bounded below. Take $\beta_A = \inf A$ and $\alpha_B = \sup B$. Then we have $\beta_A \leq a$ for $a \in A$, and $\alpha_B \geq b$ for $b \in B$. This last sentence is equivalent to having $-2\alpha_B \leq -2b$ for all $b \in B$. Adding these shows that $\beta_A - 2\alpha_B$ is a lowerbound for D.

Lets define the following:

Definition 1.2.2 — Convergence of a sequence. A sequence (s_n) converges to a value s if and only if for every $\varepsilon > 0$ there exists a number M so that if $n > M$ then $|s_n - s| < \varepsilon$.

Exercise 1.3 Using definition 1.2.3, show that

$$s_n = \left(-\frac{7}{n^3 + 2}\right)$$

converges to 0. ■

Proof. Let $\varepsilon > 0$, then choose $M = 7/\varepsilon$. Then for all $n > M$ we have

$$n > \frac{7}{\varepsilon}$$

which implies

$$\varepsilon > \frac{7}{n} \geq \frac{7}{n^3} > \frac{7}{n^3+2} = \left| -\frac{7}{n^3+2} \right|$$

that holds for all $n \geq 1$. Therefore

$$\left| -\frac{7}{n^3+2} - 0 \right| < \varepsilon$$

■

Exercise 1.4 Write the sequence

$$\sqrt{2} \quad , \quad \sqrt{2+7\sqrt{2}} \quad , \quad \sqrt{2+7\sqrt{2+7\sqrt{2}}} \quad , \quad \sqrt{2+7\sqrt{2+7\sqrt{2+7\sqrt{2}}}} \cdots$$

recursively, to demonstrate that the sequence satisfies the conditions to apply the Monotone Convergence Theorem.

Then find the limit of the sequence. ■

We write this sequence recursively with seed $s_1 = \sqrt{2}$ and define $s_{n+1} = \sqrt{2+7s_n}$. Now recall:

Theorem 1.2.1 If a sequence is monotone-increasing, and bounded above then it converges.

We will prove that s_n defined as we have done, converges. Looking at the first few terms means we can guess an upperbound of ≈ 8. Also note that $s_2 \approx 3.45$

Proof. See that s_1 is bounded above by 8. Then we have

$$\begin{aligned} s_n &< 8 \\ 7s_n &< 56 \\ 2+7s_n &< 58 < 64 \\ \sqrt{2+7s_n} &< 8 \\ s_{n+1} &< 8 \end{aligned}$$

thus, by induction we have s_n is bounded above by 8. Now we will show that s_n is an increasing sequence. Our calculation shows that $s_1 < s_2$. This is our base case. Now assume that $s_k < s_{k+1}$ then we have

$$\begin{aligned} s_{k+1} &> s_k \\ 2+7s_{k+1} &> 2+7s_k \\ \sqrt{2+7s_{k+1}} &> \sqrt{2+7s_k} \\ s_{k+2} &> s_{k+1} \end{aligned}$$

thus, s_n is monotone increasing. Applying theorem 1.2.1 shows that s_n converges. ■

Now let's find the limit of the sequence. First let $L = \lim_{n\to\infty} s_n$. Then $s_{n+1} = \sqrt{2+7s_n}$, after taking the limit of both sides, gives

$$\begin{aligned} L &= \sqrt{2+7L} \\ L^2 - 7L - 2 &= 0 \\ L &= \frac{7+\sqrt{57}}{2} \end{aligned}$$

which is the limit, which can be approximated as $L \approx 7.34$. (Now technically there is a negative root, but obviously we don't choose that one).

Exercise 1.5 Using the presented definition, show that the following limit does **not** exist:

$$\lim_{x\to 3} \frac{|x-3|}{3-x}$$

■

Definition 1.2.3 We define the limit

$$\lim_{x\to a} f(x) = b$$

to mean that if (x_n) is a sequence in $\mathrm{Dom}_f \backslash \{a\}$ that converges to a, then the sequence $f(x_n)$ converges to b.

Proof. For this problem, we can prove this via contradiction. First see that $\mathrm{Dom}_f = \mathbb{R}\backslash\{3\}$ in this case. We produce two sequences that approach 3:

$$a_n := \frac{1}{n} + 3 \qquad b_n := 3 - \frac{1}{n} \qquad (a_n), (b_n) \subset \mathbb{R}\backslash\{3\}$$

since these are both sequences in Dom_f then we would expect $f(a_n) = f(b_n)$. We see after substituting a_n and b_n that this is not the case:

$$\begin{aligned} f(a_n) &= \frac{|\frac{1}{n} + 3 - 3|}{3 - \frac{1}{n} - 3} \\ &= -1 \\ f(b_n) &= \frac{|3 - \frac{1}{n} - 3|}{3 - 3 + \frac{1}{n}} \\ &= 1 \end{aligned}$$

To conclude, we made two sequences $a_n \to 3$, $a_n \neq 3$ and $b_n \to 3$, $b_n \neq 3$. By our definition of limits, if $f(a_n)$ existed, it would have to equal $f(b_n)$. Since we showed $f(a_n) \neq f(b_n)$ this is in contradiction of our definition and thus no limit can exist. ■

Exercise 1.6 Prove (using the following definition) that

$$\lim_{x\to 2} \frac{x^2+x+2}{x+2} = 2$$

■

Definition 1.2.4 — Limit of a function. We say $\lim_{x\to a} f(x) = b$, where $a \in \mathbb{R}$ if the following holds:

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{such that if} \quad 0 < |x-a| < \delta, \text{then} \quad |f(x) - b| < \varepsilon$$

If there does exist $b \in \mathbb{R}$ such that the above holds, then we say $\lim_{x\to a} f(x)$ **exists**.

Let's do some scratch work. For some $\varepsilon > 0$ we wish to find some δ so that $0 < |x-2| < \delta$ implies $|f(x) - 2| < \varepsilon$. Obviously we need to choose this δ based on ε. Let's first simplify what we really want:

$$\begin{aligned}|f(x)-2| &= \left|\frac{x^2+x+2}{x+2}-2\right| \\ &= \left|\frac{x^2-x-2}{x+2}\right| \\ &= \left|\frac{(x-2)(x+1)}{x+2}\right| \\ &= |x-2|\frac{|x+1|}{|x+2|}\end{aligned}$$

which we want to eventually be less than ε. We can already control $|x-2|$ by δ. We now need to bound $|x+1|$ by something **above**, and $|x+2|$ by something **below**. Say we let $\delta = 1$, then we would have

$$\begin{aligned}|x-2| &< \delta \\ -\delta < x-2 &< \delta \\ -1 < x-2 &< 1\end{aligned}$$

which we can use to get both of the following:

$$2 < x+1 < 4 \qquad 3 < x+2 < 5$$

meaning we can bound $|x+1|$ above by 4, and we can bound $|x+2|$ below by 3. In summary we have

$$|f(x)-2| = |x-2|\frac{|x+1|}{|x+2|} < \delta\frac{4}{3}$$

Simply choose $\delta = \varepsilon/4$ for this to work. But remember we had to let $\delta \leq 1$ for all this to work, so if we want both to hold we choose $\delta = \min\{1, \frac{\varepsilon}{4}\}$. This gives us enough intuition for a proof:

Proof. Let $\varepsilon > 0$. Then choose $\delta = \min\{1, \frac{\varepsilon}{4}\}$. Then $0 < |x-2| < \delta$ can imply $|x+1| < 4$ and $|x+2| > 3$. Therefore

$$|f(x)-2| = |x-2|\frac{|x+1|}{|x+2|} < \delta\frac{4}{3} = \frac{\varepsilon}{3} < \varepsilon$$

■

Exercise 1.7 First construct a rigorous definition in the-style of ε, δ, for the limit:

$$\lim_{x\to\infty} f(x) = -\infty$$

then use this definition to show that

$$\lim_{x\to\infty} \frac{-2x}{\sqrt{4x-3}} = -\infty$$

■

We basically want something along these lines: Give me any negative number, then I will be able to find a point in the sequence where $f(x)$ after that point is less than that number you gave. Formally:

Definition 1.2.5 — Negative Infinity Limit. A function $f(x) \to -\infty$ as $x \to \infty$, if for every $K > 0$ we can find some N so that for all $x \geq N$ we will have

$$f(x) < -K$$

We can now use this definition like so:

Proof. Let $K > 0$. Then we choose $N = \max\{1, K^2\}$. This is to ensure that for any $x \geq N$, we will have both $x \geq 1$ AND $x \geq K^2$. Because we have $x \geq 1$ we can deduce

$$4x - 3 \leq 4x \quad \Longrightarrow \quad \sqrt{4x-3} \leq 2\sqrt{x}$$

and because we also have $x \geq K^2$, we can show

$$\begin{aligned} x &\geq K^2 \\ \sqrt{x} &\geq K \\ \frac{2x}{2\sqrt{x}} &\geq K \\ \frac{2x}{\sqrt{4x-3}} > \frac{2x}{2\sqrt{x}} &\geq K \end{aligned}$$

We can justify taking the positive square root of K since $K > 0$. Taking the ends of that last line, we can show

$$\frac{2x}{\sqrt{4x-3}} > K \quad \Longrightarrow \quad \frac{-2x}{\sqrt{4x-3}} < -K$$

which is exactly what we want to show. ■

2. Differentiability

> I became insane, with long intervals of horrible sanity.
>
> *Edgar Allan Poe*

2.1 Structure of these notes

This chapter builds the core tools used later in real and complex analysis by working from first principles. It starts with epsilon delta proofs of continuity and limits. It then studies differentiability, including piecewise examples. It uses Taylor polynomials with a clear error bound. It finishes with key complex analysis ideas like the Poisson kernel, the Cauchy Riemann equations, and harmonic conjugates.

2.2 Exercises

Definition 2.2.1 A function f being continuous at an accumulation point $a \in D_f$ means that for any $\varepsilon > 0$, there exists a $\delta > 0$ so that if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Exercise 2.1 Consider the function $f(x) = \sqrt[5]{x}$. Use the definition to prove that

1. f is continuous at $x = 0$.
2. f is continuous at $x = 7$.

Hint: the identity $a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$ may be helpful ■

Let's first do some scratch work. Consider the case $x = 0$. The domain of $f(x)$ is simply all of $\mathbb{R}$. So for $0 \in \mathbb{R}$, and some $\varepsilon > 0$, we need to find some $\delta > 0$ such that if $|x| < \delta$, then we will have $|\sqrt[5]{x}| < \varepsilon$. One might think that $|\sqrt[5]{x}| < |x|$ motivates choosing $\delta = \varepsilon$. But see that this is actually false for $|x| < 1$. Instead, we can adapt this thinking and instead choose $\delta = \varepsilon^5$.

Proof. Let $\varepsilon > 0$. Then choose $\delta = \varepsilon^5$. Since ε is positive, we know $\delta > 0$ as well. Now assume $|x - 0| < \delta$. Then

$$|x-0| < \varepsilon^5 \qquad \text{therefore} \qquad |x|^{\frac{1}{5}} < \varepsilon$$

and we have $|\sqrt[5]{x}| < \varepsilon$. What we have shown is that $|\sqrt[5]{x} - 0| < \varepsilon$ and therefore

$$|f(x) - f(0)| < \varepsilon$$

completing the proof. This shows f is continuous at $x = 0$. ■

We now move on to (2). To demonstrate $x^{\frac{1}{5}}$ is continuous at $x = 7$, we will need to show that fro any $\varepsilon > 0$, we can find some $\delta > 0$ such that $|x-7| < \delta$ implies $|f(x) - f(7)| < \varepsilon$. The first task is to simplify the expression $|f(x) - f(7)| = |x^{\frac{1}{5}} - 7^{\frac{1}{5}}|$. If we let $a = x^{\frac{1}{5}}$ and $b = 7^{\frac{1}{5}}$ then by our hint:

$$x - 7 = \left(x^{1/5} - 7^{1/5}\right)\left((x^{1/5})^4 + (x^{1/5})^3 7^{1/5} + (x^{1/5})^2(7^{1/5})^2 + x^{1/5}(7^{1/5})^3 + (7^{1/5})^4\right)$$

and thus:

$$|x^{1/5} - 7^{1/5}| = \frac{|x-7|}{\left|(x^{1/5})^4 + (x^{1/5})^3 7^{1/5} + (x^{1/5})^2(7^{1/5})^2 + x^{1/5}(7^{1/5})^3 + (7^{1/5})^4\right|}$$

Suppose we choose $\delta = 1$. Then $|x-7| < 1$ and therefore $x \in (6,8)$. Remember we want to bound denominator below. By our choice of δ the smallest x can be is 6, so then $x^{1/5} > 6^{1/5}$. This means the lowest $x^{1/5}$ can be is $6^{1/5}$. Also see that $7^{1/5} > 6^{1/5}$. Since every term in the denominator has at least 4 factors, then each term is at least $(6^{1/5})^4$. So our lowerbound for the denominator is

$$\text{number of terms} \cdot (6^{1/5})^4 = 5 \cdot 6^{\frac{4}{5}}$$

In summary, we have a lower bound for the denominator by choosing $\delta = 1$. Thus we can establish:

$$|x^{1/5} - 7^{1/5}| = \frac{|x-7|}{\text{horrid looking denominator}} \leq \frac{|x-7|}{5 \cdot 6^{4/5}}$$

So to get our desired inequality we choose $\delta = 5 \cdot 6^{4/5}\varepsilon$. We can proceed with a proof:

Proof. We will show $f(x) := \sqrt[5]{x}$ is continuous at $x = 7$. Let $\varepsilon > 0$. Then choose $\delta = \min\{1, 5 \cdot 6^{4/5}\varepsilon\}$. Now suppose $|x-7| < \delta$. Then we would have $|x-7| < 1$ which allows us to establish

$$|f(x) - f(7)| = |x^{1/5} - 7^{1/5}| < \frac{|x-7|}{5 \cdot 6^{4/5}} < \varepsilon$$

■

R Of course a tighter lower bound for the denominator can be found, however the reader may find it overwhelming to follow such computations, when a sufficient alternative exists.

Exercise 2.2 Use the Intermediate Value Theorem to show $\sqrt[4]{3}$ exists. ■

Theorem 2.2.1 If a function f is continuous on a closed interval $[a,b]$ and $f(a) = f(b)$, then for any number N between $f(a)$ and $f(b)$ there exists $c \in [a,b]$ so that $f(c) = N$.

Proof. We need to show there exists a c such that $c^4 = 3$. Set $f(x) = x^4$. Then $f(x)$ is continuous on $[1,2]$ with $f(1) = 1$ and $f(2) = 16$. Then set $N = f(c) = 3$ which has $f(1) < 3 < f(2)$. By IVT there exists some $c \in [1,2]$ such that $f(c) = c^4 = 3$. Therefore $c = \sqrt[4]{3}$ exists. ■

Exercise 2.3 For each of the following functions determine whether the function is differentiable at the specified value of x:

1.
$$f(x) = \begin{cases} 2x+2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$
for $x = 1$.

2.

$$f(x) = \begin{cases} x^4 - x^3 + 2x - 2 & x \leq 1 \\ \frac{3}{2}(x^2 - 1) & x > 1 \end{cases}$$

at $x = 1$.

We will use the following definition of a derivative:

Definition 2.2.2 A function f is differentiable at an accumulation point $a \in D_f$ when the limit

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite.

It is important to remember that all differentiable functions are continuous, *but not all continuous functions are differentiable.* Let's examine (1). Here we find $f(1) = 2$. Now lets find the limit of $f(x)$ as $x \to 1$. That would be $\lim_{x \to 1} f(2x+2)$. This, we find is $\lim_{x \to 1} f(2x+2) = 4$. Since $\lim_{x \to 1} f(x) \neq f(1)$ then *the function is not continuous* at $a = 1$. Therefore, it is not differentiable at $a = 1$.
As for (2), one can find that the limit at $x = 1$ is indeed continuous, so it is possible that this function is differentiable at the point $a = 1$. We will first calculate the derivative on the right side of the point, and check if it matches the derivative on the left side. See that if we choose $x > 1$, then

$$\begin{aligned} f(x) &= \tfrac{3}{2}(x^2 - 1) \\ f'(1) &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \to 1} \frac{\frac{3}{2}(x^2 - 1) - 0}{x - 1} \\ &= \lim_{x \to 1} \frac{\frac{3}{2}(x+1)(x-1)}{x - 1} \\ &= \lim_{x \to 1} \tfrac{3}{2}(x+1) \\ &= 3 \end{aligned}$$

now suppose $x \leq 1$, then we are now approaching from the left side of 1, and

$$\begin{aligned} f(x) &= x^4 - x^3 + 2x - 2 \\ f'(1) &= \lim_{x \to 1} \frac{(x^4 - x^3 + 2x - 2) - 0}{x - 1} \\ &= \lim_{x \to 1} \frac{(x^3 + 2)(x - 1)}{x - 1} \\ &= \lim_{x \to 1} x^3 + 2 \\ &= 3 \end{aligned}$$

since the left derivative is equal to the right derivative, then the derivative at the point $x = 1$ exists.

Exercise 2.4 Use the Taylor polynomial of order $n = 2$ for $f(x) = e^{x^2 - 1}$ about $x = 1$ to obtain an approximate value for $f(1.02)$ and then use Taylor's Theorem to show that this approximation gives the value of $f(1.02)$ correctly, to 4 decimal places. ■

Recall the Taylor polynomial of order 2 centered at $a = 1$ is given by

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

so we first find

$$\begin{aligned}
f(1) &= e^{1-1} = 1 \\
f'(x) &= 2xe^{x^2-1} f'(1) \\
&= 2 \\
f''(x) &= 2e^{x^2-1} + 4x^2 e^{x^2-1} \\
f''(1) &= 2 + 4 = 6
\end{aligned}$$

and after substituting we have

$$1 + 2(x-1) + 3(x-1)^2$$

now we want to use the above formula to approximate $x = 1.02$. So keeping in mind $x - 1 = .02$ in this case we will get the approximation

$$1 + 2(.02) + 3(.02)^2 = 1 + .04 + .0012 = 1.0412$$

To show this approximation is accruate to 4 decimals, we will use the following therorem:

Theorem 2.2.2 Let $P_n(x)$ be the Taylor polynomial of order n for $f(x)$ about a. Then for each $x \neq a$ in some open interval I containing a, there exists some number c between a and x such that

$$f(x) - P_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

By the above theorem there exists some number $c \in (1, 1.02)$, such that

$$f(x) - P_n(x) = \frac{(1.02-1)^3}{3!} f'''(c) = \frac{0.000008}{6} f'''(c) = 0.00000133333 \cdot f'''(c)$$

So we have some number c such that

$$f(1.02) - 1.0412 = 0.00000133333 \cdot f'''(c)$$

We calculate the third order derivative:

$$f'''(x) = \frac{d}{dx} 2e^{x^2-1} + 4x^2 e^{x^2-1} = 12e^{x^2-1}x + 8x^3 e^{x^2-1}$$

Here since $1 < c < 1.02$ and $f'''(x)$ is positive and increasing (at least on this interval) we will have

$$f'''(1) < f'''(c) < f'''(1.02) \qquad \text{implies} \qquad 20 < f'''(c) < 21.5841 < 22$$

Therefore:

$$\begin{aligned}
20 <& f'''(c) < 22 \\
0.00000133333 \cdot 20 < 0.00000133333 \cdot& f'''(c) < 0.00000133333 \cdot 22 \\
0.0000266666 <& f(1.02) - 1.0412 < 0.00002933326 \\
1.0412266666 <& f(1.02) < 1.04122933326
\end{aligned}$$

in this case, even with the sub-optimal upper bound 22, we are still accurate to 4 decimals.

Exercise 2.5 This question can be divided into 2 parts.

1. Show that for $w \neq z$,
$$\operatorname{Re}\left(\frac{w+z}{w-z}\right) = \frac{|w|^2-|z|^2}{|w-z|^2}$$
2. Let $z = re^{i\theta}$ and $w = Re^{i\phi}$, where $i \leq r < R$. The *Poisson Kernel* is defined as
$$P(w,z) := \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}$$
Prove that
$$\operatorname{Re}\left(\frac{w+z}{w-z}\right) = P(w,z)$$

Corollary 2.2.3 We will need to use the following:

1. Given some complex number z, we have $\overline{(\bar{z})} = z$.
2. Given two complex numbers w,z, we will have $\overline{(wz)} = \bar{w}\bar{z}$.
3. For two complex numbers w,z, we have $\overline{(w\bar{z})} = \bar{z}w$.
4. For some complex number z we have $\bar{z}-z = -2i\operatorname{Im}(z)$.

Proof. Here (1) is trivial, (2) is product rule, and (3) can be found by applying (1) and (2). Also (4) can be found by direct computation. ■

Let us simplify the first problem by multiplying by its conjgate like so:

$$\frac{w+z}{w-z} = \frac{w+z}{w-z}\cdot\frac{\bar{w}-\bar{z}}{\bar{w}-\bar{z}} = \frac{(w+z)(\bar{w}-\bar{z})}{|w-z|^2} = \frac{w\bar{w}-w\bar{z}+\bar{w}z+z\bar{z}}{|w-z|^2} = \frac{|w|^2-|z|^2+w\bar{z}-\bar{w}z}{|w-z|^2}$$

Now since anything of the form $|\text{something}|^2$ is a real number we have

$$\operatorname{Re}\left(\frac{w+z}{w-z}\right) = \operatorname{Re}\left(\frac{|w|^2-|z|^2}{|w-z|^2}+\frac{w\bar{z}-\bar{w}z}{|w-z|^2}\right) = \frac{|w|^2-|z|^2}{|w-z|^2}+\operatorname{Re}\left(\frac{w\bar{z}-\bar{w}z}{|w-z|^2}\right)$$

Let's look at the terms $w\bar{z}-\bar{w}z$. Notice that $w\bar{z}$ is the conjugate of $\bar{w}z$ by Corollary 2.2.2 (3). Then by Corollary 2.2.2 (4) we have $w\bar{z}-\bar{w}z = -2i\operatorname{Im}(\bar{w}z)$. This means there is no real part of $w\bar{z}-\bar{w}z$. In total we get:

$$\operatorname{Re}\left(\frac{w+z}{w-z}\right) = \frac{|w|^2-|z|^2}{|w-z|^2}+\frac{\operatorname{Re}(w\bar{z}-\bar{w}z)}{|w-z|^2} = \frac{|w|^2-|z|^2}{|w-z|^2}+\frac{0}{|w-z|^2} = \frac{|w|^2-|z|^2}{|w-z|^2}$$

which proves the proposition.

Moving on to (2). First we observe the following:

$$\begin{aligned}
|w|^2 &= w\bar{w} = (Re^{i\phi})(Re^{-i\phi}) = R^2(e^{\phi(i-i)}) = R^2\\
|z|^2 &= z\bar{z} = (re^{i\theta})(re^{-i\theta}) = r^2(e^{\theta(i-i)}) = r^2\\
\therefore\quad |w|^2-|z|^2 &= R^2-r^2
\end{aligned}$$

So we have substituted our values of z and w to get an expression for the numerator of Poisson

Kernel. Now we turn our attention to the denominator. This can be calculated like so:

$$\begin{aligned}|w-z|^2 &= (w-z)(\bar{w}-\bar{z}) = (Re^{i\phi}-re^{i\theta})(Re^{-i\phi}-re^{-i\theta})\\ &= (Re^{i\phi})(Re^{-i\phi})-(Re^{i\phi})(re^{-i\theta})-(re^{i\theta})(Re^{-i\phi})+(re^{i\theta})(re^{-i\theta})\\ &= R^2-Rr(e^{i(\theta-\phi)}+e^{-i(\theta-\phi)})+r^2\\ &= R^2-Rr2\cos(\theta-\phi)+r^2\end{aligned}$$

Here on the second last line we swapped the middle two terms, and the last line used Eulers formula. In summary we have shown:

$$\operatorname{Re}\left(\frac{w+z}{w-z}\right)=\frac{|w|^2-|z|^2}{|w-z|^2}=\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}=P(w,z)$$

for $z=re^{i\theta}$ and $w=Re^{i\theta}$. This finishes the proof.

Exercise 2.6 For each of the following choices of f use the definition of a limit to obtain $\lim_{z\to 0} f(z)$ or prove that the limit does not exist:

1. $$\frac{1}{z}+\frac{1}{z(z-1)}$$
2. $$\frac{\bar{z}}{z}$$
3. $$\frac{\bar{z}-1}{|z|-1}$$
4. $$\frac{\operatorname{Re}(z)\operatorname{Im}(z)}{|z|}$$

■

First let us recall the definition of a limit, for complex numbers. A foundation of limits in the real numbers is based on writing inequalties such as $x<y$. We cannot do this for complex numbers. We can, however, write $|x|<|y|$ for $x,y\in\mathbb{C}$. We only have to make a small change:

Definition 2.2.3 — Complex Limit. Suppose that a complex function f is defined in a deleted neighborhood of z_0 and suppose that w_0 is a complex number. The limit of f as z tends to z_0 exists and is equal to w_0, written as $\lim_{z\to z_0} f(z)=w_0$ if the following holds:

$$\forall\varepsilon>0,\exists\delta>0 \qquad \text{such that} \qquad |f(z)-w_0|<\varepsilon \qquad \text{whenever} \qquad 0<|z-z_0|<\delta$$

For the first problem, we can simplify like so:

$$\frac{1}{z}+\frac{1}{z(z-1)}=\frac{(z-1)}{z(z-1)}+\frac{1}{z(z-1)}=\frac{z}{z(z-1)}=\frac{1}{z-1}$$

and we are allowed to cancel z from the denominater since we are only considering $z\to 0$ and not $z=0$. We know by substitution the answer is simply $\lim_{z\to 0} f(z)=-1$, but we will prove this using the definition. Take some $\varepsilon>0$, we will show there exists $\delta>0$ such that $0<|z|<\delta$ implies $|f(z)-(-1)|<\varepsilon$. First see that

$$|f(z)+1|=\left|\frac{1}{z-1}+1\right|=\frac{|z|}{|z-1|}$$

remember with fractions like this, we want to bound the numerator $|z|$ above, and the denominator $|z-1|$ below. One can attempt to bound $|z|$ above by choosing $\delta = 1$, but the reader will see this does not work. Instead we choose $\delta = \frac{1}{2}$. Note that $|z-1|$ can be bounded below by inverse triangle inequality:

$$|z-1| \geq ||z|-|1|| = ||z|-1| = 1-|z|$$

and since $|z| < \frac{1}{2}$ we have $1-|z| > 1-\frac{1}{2}$ and thus $1-|z| > \frac{1}{2}$. Therefore $\frac{1}{|z-1|} < 2$. This gives us

$$\frac{|z|}{|z-1|} < 2|z|$$

We also need to let $|z| < \frac{\varepsilon}{2}$ to continue. Keeping in mind we needed $\delta = \frac{1}{2}$ for all this to work, we are able to present the following proof.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$. Now suppose $0 < |z-0| < \delta$. Then we would have $|z| < \frac{1}{2}$. This would imply $\frac{1}{|z-1|} < 2$, and therefore

$$|f(z)-(-1)| = \frac{|z|}{|z-1|} < 2|z|$$

Becuase $|z| < \frac{\varepsilon}{2}$ we have the conclusion $|f(z)-(-1)| < \varepsilon$. This shows

$$\lim_{z\to 0}\left(\frac{1}{z}+\frac{1}{z(z-1)}\right) = -1$$

. ■

Now we move on to (2). We will show this limit $\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist. If the limit truly did exist, then we can let the point $z = x+yi$ approach $z = 0$ from any direction and expect the same result. Suppose z approaches the origin on the real line, then it will be of the form $z = x+0i$. Then

$$f(z) = \frac{x-0i}{x+0i} = \frac{x}{x} = 1$$

in which case $f(z) \to 1$ as $z \to 0$. But suppose z approaches the origin from the imaginary axis, then it is of the form $z = 0+yi$. This would mean

$$f(z) = \frac{0-yi}{0+yi} = \frac{-yi}{yi} = -1$$

and thus as $z \to 0$ we would have $f(z) \to -1$. This is a contradiction.

Now we move on to (3). Clearly by substitution we will have the limit is equal to 1. We will prove this with the definition. As usual, we assume $\varepsilon > 0$, and we want to show that there exists such a $\delta > 0$ such that $0 < |z| < \delta$ implies $|f(z)-1| < \varepsilon$. We can simplify as so:

$$|f(z)-1| = \left|\frac{\bar{z}-1}{|z|-1}-1\right| = \left|\frac{\bar{z}-|z|}{|z|-1}\right| = \frac{|\bar{z}-|z||}{||z|-1|}$$

We will first bound the numerator *above*. First see that $|\bar{z}-|z|| = |\bar{z}+(-|z|)| \leq |\bar{z}|+|-|z|| = |\bar{z}|+|z|$ by triangle inequality. And since $|\bar{z}| = |z|$ that reduces further, and we have

$$|\bar{z}-|z|| \leq 2|z|$$

Great. So we have bounded the numerator above, we need to bound the denominator *below*. It's a good idea to bound $|z|$ above by $\delta = \frac{1}{2}$. This way, we have $|z|-1<0$ by obviousness. This allows us to say $||z|-1| = 1-|z|$. But remember $-|z| > -\frac{1}{2}$. So $1-|z| > \frac{1}{2}$. In summary, we have bounded the denominator below by $\frac{1}{2}$. Using these bounds:

$$|f(z)-1| = \frac{|\bar{z}-|z||}{||z|-1|} \leq \frac{2|z|}{\frac{1}{2}} = 4|z|$$

so as usual we also want to bound $|z| < \frac{\varepsilon}{4}$. Keeping in mind we had to bound $|z|$ above by $\frac{1}{2}$ for all this to work, we can present a formal proof.

Proof. We will show the limit of the function as $z \to 0$ is 1. Given some $\varepsilon > 0$ choose $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{4}\right\}$. Suppose $0 < |z-0| < \delta$. Then

$$|f(z)-1| = \frac{|\bar{z}-|z||}{||z|-1|} \leq \frac{2|z|}{\frac{1}{2}} = 4|z| < \varepsilon$$

which is true by triangle inequality and the upper bound $|z| < \frac{1}{2}$. By definition 2.2.3 we have proven the proposition. ■

We move on to (4) at last. We write z of the form $z = x+yi$. Picture taking this dot approaching the origin. Then what we want to see is if $xy/|z|$ approaches 0. Suppose $|z| = r$ which is the distance from the origin to the point. Then obviously $x \leq r$ and $y \leq r$. We only need to observe the absolute value of the limit in this case, so we have

$$\left|\frac{\mathrm{Re}(z)\mathrm{Im(z)}}{|z|}\right| = \left|\frac{xy}{r}\right| = \frac{|x||y|}{r} \leq \frac{r^2}{r} = r$$

and since the length r goes to 0 then it tells us the limit of our function goes to 0 as well. We will now formally prove this.

Proof. Write $z = x+yi$. Then $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$. If we write $|z| = \sqrt{x^2+y^2} = r$, then $|x| \leq r$ and $|y| \leq r$. Therefore

$$\frac{\mathrm{Re}(z)\mathrm{Im(z)}}{|z|} = \frac{xy}{r}$$

Let $\varepsilon > 0$, and choose $\delta = \varepsilon$. Then when $0 < |z| < \delta$ then we have

$$|f(z)-0| = \left|\frac{\mathrm{Re}(z)\mathrm{Im(z)}}{|z|}\right| = \left|\frac{xy}{r}\right| = \frac{|x||y|}{r} \leq \frac{r^2}{r} = r$$

and since $r = |z|$ and $|z| < \delta = \varepsilon$ in summary we have

$$|f(z)-0| < \varepsilon$$

completing the proof. ■

Exercise 2.7 This last question is divided into two parts:

1. Use the Cauchy-Riemann equations to determine where the function $g(z) = |z|^2$ is differentiable.
2. Suppose that v is the harmonic conjugate of u in a domain $\Omega \subseteq \mathbb{C}$. Show that the function uv must also be harmonic in Ω.

■

Let us first recall the Cauchy-Riemann theorem.

Theorem 2.2.4 — Cauchy-Riemann. A complex function $f(z) = U(x,y) + V(x,y)i$ has a complex derivative $f'(z)$ if and only if its real and imaginary part are continuously differentiable and satisfy the Cauchy-Riemann equations:

$$\begin{cases} U_x = V_y \\ U_y = -V_x \end{cases}$$

and the complex derivative is given by

$$f'(z) = U_x + iV_x = V_y - iU_i$$

In this case our function can be viewed as $g(z) = |z|^2 = x^2 + y^2$. We have $U(x,y) = x^2 + y^2$ and no imaginary part so $V(x,y) = 0$. Thus $U_x = 2x$ and $U_y = 2y$. Also $V_x = V_y = 0$. Applying the equations, we want

$$\begin{cases} 2x = 0 \\ 2y = -0 \end{cases}$$

which imply $x = y = 0$. Therefore the point where a complex derivative can be found is at $(x,y) = 0$ or $z = 0$.

Lastly, we move on to (2). Here is what we know; v is the harmonic conjugate of u. This means $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$, and both u and v satisfy the Cauchy Riemann equations. What we want to show is that $(uv)_{xx} + (uv)_{yy} = 0$ on Ω. To do this we first calculate:

$$\begin{aligned} (uv)_x &= u_x v + u v_x \\ (uv)_y &= u_y v + u v_y \\ (uv)_{xx} &= u_{xx} v + 2u_x v_x + u v_{xx} \\ (uv)_{yy} &= u_{yy} v + 2u_y v_y + u v_{yy} \end{aligned}$$

And therefore

$$\begin{aligned} (uv)_{xx} + (uv)_{yy} &= (u_{xx} v + 2u_x v_x + u v_{xx}) + (u_{yy} v + 2u_y v_y + u v_{yy}) \\ &= (u_{xx} + u_{yy})v + (v_{xx} + v_{yy})u + 2(u_x v_x + u_y v_y) \\ &= 2(u_x v_x + u_y v_y) \end{aligned}$$

We are almost done; we still need to show this is equal to 0. By Cauchy Riemann euqations we would have $u_y = -v_x$ and $u_x = v_y$ so we would get

$$2(u_x v_x + u_y v_y) = 2(v_y v_x - v_x v_y) = 0$$

showing that uv must be harmonic on Ω.

3. Derivatives and Integration

> The essence of math is not to make simple things *complex*, but to make complicated things simple.
>
> *Stan Gudder*

3.1 Structure of these notes

We will now focus on standard techniques from complex integration parameterising circular contours, applying Cauchy's theorem and the Cauchy integral formula, locating singularities and determining when integrals vanish, and using residues to evaluate contour and real improper integrals.

3.2 Exercises

Notation 3.1. *Let $\gamma(a;r)$ denote the circle of radius $r>0$ centred at $a\in\mathbb{C}$ oriented anticlockwise*

Exercise 3.1 Let $gamma = \gamma(0;1)$. Evaluate the following integrals:

1.
$$\int_\gamma \frac{|z|^5}{z}\,dz$$

2.
$$\int_\gamma \sin\left(e^{z^2}\right)\,dz$$

3.
$$\int_\gamma \bar{z}\cos(|z|)\,dz$$

For the first one note that $|z|$ is the length of the unit circle; by definition 1. We are really integrating $1/z$ over the curve. This is the common integral $2\pi i$. To recap we have:

$$\int_\gamma \frac{|z|^5}{z}\,dz = \int_\gamma \frac{1}{z}\,dz = 2\pi i$$

For (2) observe that e^{z^2} and $\sin(z)$ are both entire functions, and so is their composition $\sin(e^{z^2})$.

Since γ is a closed contour we have by Cauchys theorem:

$$\int_\gamma \sin\left(e^{z^2}\right)\,dz = 0$$

Lastly for (3), we recall that $z\bar{z} = |z|^2$. Since $|z| = 1$ on γ we then have $\bar{z} = 1/z$. Also we now have $\cos(|z|) = \cos(1)$. Our integral can be evaulated as follows:

$$\int_\gamma \bar{z}\cos(|z|)\,dz = \int_\gamma \frac{1}{z}\cos(1)\,dz = \cos(1)\int_\gamma \frac{1}{z}\,dz = \cos(1)\cdot 2\pi i$$

Exercise 3.2 Let $\gamma = \gamma(0;1)$, and let $\alpha \in \mathbb{R}$ be a constant such that $0 < \alpha < 1$.

1. Use the Cauchy integral formula to evaluate

$$I_\alpha = \int_\gamma \frac{dz}{i(1-\alpha z)(z-\alpha)}$$

2. Use the definition of a line integral for I_α to show that

$$\int_0^{2\pi} \frac{d\varphi}{1-2\alpha\cos\varphi+\alpha^2} = \frac{2\pi}{1-\alpha^2}$$

■

Let us first manipulate I_α. See that

$$\frac{1}{i(1-\alpha z)(z-\alpha)} = -\frac{i}{(1-\alpha z)(z-\alpha)} = -\frac{\frac{i}{1-\alpha z}}{z-\alpha}$$

and so our integral is equivelant to

$$I_\alpha = -\int_\gamma \frac{\frac{i}{1-\alpha z}}{z-\alpha}\,dz$$

we can apply Cauchy's integral formula with $f(z) = \frac{i}{1-\alpha z}$. This means $I_\alpha = -2\pi i f(\alpha)$. We have therefore calculated:

$$I_\alpha = -2\pi i\frac{i}{1-\alpha^2} = \frac{2\pi}{1-\alpha^2}$$

Lets move on to (2). Here the parameterization for $\gamma(0;1)$ is given by $\gamma(\varphi) := e^{i\varphi}$ for $0 \le \varphi \le 2\pi$. Here $dz = ie^{i\varphi}\,d\varphi$ so we have

$$I_\alpha = \int_0^{2\pi} \frac{z}{(1-\alpha z)(z-\alpha)}\,d\varphi = \int_0^{2\pi} \frac{e^{i\varphi}}{(1-\alpha e^{i\varphi})(e^{i\varphi}-\alpha)}\,d\varphi = \int_0^{2\pi} \frac{e^{i\varphi}}{(1-\alpha e^{i\varphi})e^{i\varphi}(1-\alpha e^{-i\varphi})}\,d\varphi$$

Now we can cancel the $e^{i\varphi}$ and do some manipulations of the denominator like so:

$$\int_0^{2\pi} \frac{1}{(1-\alpha e^{i\varphi})(1-\alpha e^{-i\varphi})}\,d\varphi = \int_0^{2\pi} \frac{1}{1-\alpha(e^{i\varphi}+e^{-i\varphi})+\alpha^2}\,d\varphi = \int_0^{2\pi} \frac{1}{1-2\alpha\cos\varphi+\alpha^2}\,d\varphi$$

What we have shown is that

$$I_\alpha = \int_\gamma \frac{dz}{i(1-\alpha z)(z-\alpha)} = \int_0^{2\pi} \frac{d\varphi}{1-2\alpha\cos\varphi+\alpha^2}$$

And since by the previous question we found $I_\alpha = 2\pi/(1-\alpha^2)$ we have finished for (2).

Exercise 3.3 Prove that each of the following integrals is zero:

1. $$\int_{\gamma(2i;2)} \frac{1}{1-iz}dz$$
2. $$\int_{\gamma(1;2)} \frac{1}{(z-1)^3}dz$$
3. $$\int_{\gamma(1;1)} \frac{1}{z^4+16}dz$$
4. $$\int_{\gamma(0;1)} \frac{1}{1+e^{2z}}dz$$

■

For the firt one, observe a singularity appears at $z=-i$, for this is when the denominator becomes $1-iz=1-1=0$. This singularity is obviously not in the contour since the closest point from $\gamma(2i;2)$ to the point $-i$ is actually the point $0+0i$. This means the contour does not cover that singularity, and thus we can apply Cauchy-Goursat Theorem to see

$$\int_{\gamma(2i;2)} \frac{1}{1-iz}dz = 0$$

For (2) we simply use Cauchy Integral formula again to get

$$\int_{\gamma(1;2)} \frac{1}{(z-1)^3}dz = \frac{2\pi i}{2!}f''(1) = \pi i f''(1)$$

and since $f(z)=1$ then $f'(z)=f''(z)=0$ and so we get

$$\int_{\gamma(1;2)} \frac{1}{(z-1)^3}dz = 0$$

For (3) we first need to find the singualrities of the function we are integrating. It would be lovely if we had the denominator z^4-16, but alas we are left to fiddle with z^4+16. Still, we can find the singualrities by solving for $z^4+16=0$, or $z^4=-16=16e^{\pi i}$. Taking the 4th root gives

$$z = 2e^{i(\pi+2\pi k)/4} \quad , \quad k\in\{0,1,2,3\}$$

We don't need to find these explicity, just look at their modulus. They are all 2 units away from the origin. This means it's impossible for them to be contained within our contour. There is one exception. The only time when our contour is 2 units away from the origin is at the point $z=2$. But 2 is not one of our roots since $2^4+16\neq 0$.

Now lets take a look at this last problem. As before, we find our singularity at the point that solves $e^{2z}=-1$. The first root is where $z=\frac{\pi i}{2}$. But since this is periodic there are actually an infinite amount of solutions given by $z=\frac{\pi i}{2}+\pi i k$ for $k\in\mathbb{Z}$. We need to check if any of these are contaied within the unit circle. The smallest possible values for the singularities are at $k=0$ and $k=1$. Here at $k=0$ we get $z=\frac{\pi}{2}i$. This is more than one unit away from the origin on the i axis. At $k=1$ we have $z=\frac{\pi i}{2}-\frac{2\pi i}{2}=-\frac{\pi}{2}i$. This again is now more than one unit below the origin, on the $i-$axis. Since none of the singularities are contained within the contour we have

$$\int_{\gamma(0;1)} \frac{1}{1+e^{2z}}dz = 0$$

Exercise 3.4 Suppose that f is a non constant function that is analytic in the bounded domain Ω and continuous on the boundary of Ω. Use the function e^f to deduce that $\mathrm{Re}(f)$ must achieve its maximum on the boundary of Ω and not in Ω. ■

Proof. We may simply use the Maximum Modulus Theorem. First see that since e^z is entire, and f is analytic in Ω, then e^f is also analytic. Since f is analytic and non-constant + continuous on $\bar{\Omega}$, by maximum modulus theorem $|e^f| = e^{\mathrm{Re}(f)}$ has a maximum on $\partial\Omega$. Since e^z is a strictly increasing function, $\mathrm{Re}(f)$ is maximised on $\partial\Omega$. ■

Exercise 3.5 Determine the values of z for which the following series converge absolutely:

1. $$\sum_{n=1}^{\infty} \left(\frac{z-1}{z+1}\right)^n$$
2. $$\sum_{n=1}^{\infty} \frac{1}{n^2}\left(z^n + z^{-n}\right)$$
3. $$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$$

■

For (1) we are given a geometric series with common ratio $\frac{z-1}{z+1}$. Remember this only converges when $\left|\frac{z-1}{z+1}\right| < 1$. This is equivelant to $|z-1| < |z+1|$. If we set $z = x+yi$ we have the following:

$$\begin{aligned}
|z-1| &< |z+1| \\
|x+yi-1| &< |x+yi+1| \\
|(x-1)+yi| &< |(x+1)+yi| \\
(x-1)^2+y^2 &< (x+1)^2+y^2 \\
x^2-2x+1 &< x^2+2x+1 \\
-4x &< 0 \\
x &> 0
\end{aligned}$$

So (1) only converges when $\mathrm{Re}(z) \geq 0$.
As for (2), we want the series of absolute terms to converge, so using triangle inequality we want:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}|z^n| + |z^{-n}| = \sum_{n=1}^{\infty} \frac{|z^n|}{n^2} + \sum_{n=1}^{\infty} \frac{1}{|z|^n n^2}$$

If we can show the two component series above each converge, then by Comparison test we will know (2) also converges absolutley. Lets examine the first component $\sum_{n=1}^{\infty} \frac{|z^n|}{n^2}$. If we use ratio test we see

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{|z|^{n+1}/(n+1)^2}{|z|^n/n^2}\right| = |z| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = |z|$$

And by ratio test we see this first component will only converge when $|z| < 1$. In the case $|z| = 1$, we simply have the series $\sum_{n=1}^{\infty} 1/n^2$ which converges. Overall, the first component will converge when $|z| \leq 1$. As for the second component $\sum_{n=1}^{\infty} \frac{1}{|z|^n n^2}$, applying ratio test shows this converges only when $z > 1$. Since the case $|z| = 1$ gives the same convergent series as last time, this second component only converges when $|z| \geq 1$. Clearly, the only time when both series converge is when

$|z| \geq 1$ and $|z| \leq 1$, which can only happen on the unit circle. So (2) only converges when $|z| = 1$.

Lastly we look at (3). We have to divide by cases for $|z|$.

- If $|z| < 1$. Suppose $|z| < 1$, then as $n \to \infty$ we have $|z|^n \to 0$, and thus $|1-z^n| \to 1$ as $n \to \infty$. Now remember our goal is to bound the *denominator* $|1-z^n|$ **below**. Becuase the limit of $|1-z^n| = 1$, by definition there exits some N such that for all $n \geq N$ we have

  $$||1-z^n|-1| < \frac{1}{2}$$

  This was done by choosing $\varepsilon = 1/2$. Now manipulating the expression we have

  $$\begin{aligned} ||1-z^n|-1| &< \frac{1}{2} \\ -\frac{1}{2} < |1-z^n|-1 &< \frac{1}{2} \\ \frac{1}{2} < |1-z^n| &< \frac{3}{2} \end{aligned}$$

  which gives us the lower bound we want, of $\frac{1}{2}$. So far we know there is some N such that for all $n \geq N$ we have $1/2 < |1-z^n|$ and therefore $(|1-z^n|)^{-1} < 2$, for all $n \geq N$. We have achieved our goal of getting an upper bound for our terms of the sequence:

  $$\left|\frac{z^n}{1-z^n}\right| = |z|^n \cdot \frac{1}{|1-z^n|} < 2|z|^n$$

  and since $2\sum_{n=1}^{\infty}|z|^n$ converges becuase $|z| < 1$, then by comparion test we know the series $\sum_{n=1}^{\infty} z^n(1-z^n)^{-1}$ converges, for $|z| < 1$.
- If $|z| > 1$. Suppose $|z| > 1$, then we will use divergence test to show the series diverges. That is, we will show the terms of the sequence do **not** approach 0 as $n \to \infty$, given $|z| > 1$. First see

  $$\left|\frac{z^n}{1-z^n}\right| = \left|\frac{z^n}{z^n\left(\frac{1}{z^n}-1\right)}\right| = \frac{1}{|\frac{1}{z^n}-1|}$$

  which approaches 1 since $1/|z^n| \to 0$ as $n \to \infty$. Thus, since $\lim_{n\to\infty} z^n(1-z^n)^{-1} = 1$ which is non-zero, by divergence test we know the series diverges, provided $|z| > 1$.
- If $|z| = 1$. First of all, in the case where z *equals* 1 the series isn't even defined. Now if $|z| = 1$ for something else, then first see that $|1-z^n| \leq |1|+|1| = 2$. Since $|z| = 1$ then by divergence test we would want

  $$\lim_{n\to\infty} \frac{|z|^n}{|1-z^n|} = \lim_{n\to\infty} \frac{1}{|1-z^n|} = 0$$

  So if the terms do go to 0 as $n \to \infty$ we would want $|1-z^n| \to \infty$, but this is impossible since we just bounded it above by 2! With this contradiction, we know the series diverges at $|z| = 1$ by divergence test.

Summarising what we have learned, the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$$

only converges absolutley for $|z| < 1$.

Exercise 3.6 Find the principal part of the innermost Laurent expansion about the indicated point z_0 for the functions:

1. $$\frac{1}{z^2 \sin z} \qquad \text{at} \qquad z_0 = 0$$
2. $$\frac{e^z - 1}{e^z + 1} \qquad \text{at} \qquad z_0 = \pi i$$

■

For the first one see that the only singularity occurs at $z_0 = 0$. Apart from this, $\sin(z)$ causes a singularity at $\pm\pi$, being the closest, so our innermost annulus is $\{z : 0 < |z| < \pi\}$. Using the Taylor series for $\sin(z)$ we find:

$$\begin{aligned}
\sin(z) &= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)\\
z^2\sin(z) &= z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)\\
\frac{1}{z^2\sin(z)} &= \frac{1}{z^3}\frac{1}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}\\
&= \frac{1}{z^3}\frac{z}{\sin(z)}\\
&= \frac{1}{z^2}\csc(z)\\
&= \frac{1}{z^2}\left(\frac{1}{z} + \frac{z}{6} + \cdots\right)\\
&= \frac{1}{z^3} + \frac{1}{6z} + \cdots
\end{aligned}$$

And any terms after that are of the form cz^n where n is positive. So the principal part of the innermost Laurent expansion at $z_0 = 0$ is

$$\frac{1}{z^3} + \frac{1}{6z}$$

Lets move on to (2). The first thing to notice is it looks awfully similar to our hyperbolic functions:

$$\begin{aligned}
2\sinh(z) &= e^z - e^{-z}\\
2\cosh(z) &= e^z + e^{-z}
\end{aligned}$$

We can get the above equations by factoring out $e^{z/2}$ like so:

$$\frac{e^z - 1}{e^z + 1} = \frac{e^{z/2}(e^{z/2} - e^{-z/2})}{e^{z/2}(e^{z/2} + e^{-z/2})} = \frac{2\sinh(z/2)}{2\cosh(z/2)} = \tanh\left(\frac{z}{2}\right)$$

Now we can do some substitution. Let $u = z - \pi i$, which means we can center our Laurent series at $u = 0$ forcing $z = \pi i$. This means $e^u = -e^z$, and we are able to see

$$\frac{e^z - 1}{e^z + 1} = \frac{-e^u - 1}{-e^u + 1} = \frac{e^u + 1}{e^u - 1} = \coth\left(\frac{u}{2}\right)$$

Now this honestly feels like we are going around in circles, but now we are able to use the expansion of $\coth\left(\frac{u}{2}\right)$ at $u = 0$:

$$\coth k = \frac{1}{k} + \frac{k}{3} + \cdots$$
$$\coth\frac{w}{2} = \frac{2}{w} + \frac{w}{6} + \cdots$$

Now the only negative power of w is at the term $\frac{2}{w}$. This means our principal part of the Laurent expansion $(e^z - 1)/(e^z + 1)$ at the point $z_0 = \pi i$ is equal to $\frac{2}{z-\pi i}$. Also note that this Laurent expansion centered at πi is valid only for the annulus $0 < |z - \pi i| < 2\pi$.

Exercise 3.7 Where do the following series define analytic functions?

1.
$$\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n}$$

2.
$$\sum_{n=0}^{\infty} z^{5n}$$

■

For (1) this is simply the expansion of $\ln(x+1)$ multiplied by -1. See:

$$\ln(z+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$$
$$-\ln(z+1) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n}$$

Which only converges for $|z| < 1$.

Lets move on to (2). See that this is a geometric series with ratio z^5. It will only converge when $|z^5| < 1$, or when $|z| < 1$. So it defines an analytic function on the open disk $\{z : |z| < 1\}$.

R Now techinally (1) converges on the bounday point $z = 1$ for example, it does not have an open neiborhood around 1 it is analytic on, since it is on $\partial\Omega$ by definition.

Exercise 3.8 Use the calculus of residues to evaluate the following integral:

1.
$$\int_{\gamma(0;2)} \frac{1}{(z-1)^2(z^2+1)}\,dz$$

2.
$$\int_{\gamma(0;2)} \frac{1}{1+e^{2z}}\,dz$$

■

Here we have singularities at $z = 1$ and $z = \pm i$, which are both contained within the contour. For the singularity at $z = i$, which has order 1, we calculate the residue using cover up method:

$$\operatorname{Res}(f,i) = \left.\frac{1}{(z-1)^2(z+i)}\right|_{z=i} = \frac{1}{(i-1)^2 2i} = \frac{1}{(-2i)(2i)} = \frac{1}{4}$$

and likewise for $z = -i$:

$$\operatorname{Res}(f,-i) = \left.\frac{1}{(z-1)^2(z-i)}\right|_{z=-i} = \frac{1}{(-i-1)^2(-2i)} = \frac{1}{(2i)(-2i)} = \frac{1}{4}$$

and lastly for $z = 1$, but with $m = 2$ to get

$$\begin{aligned}
\operatorname{Res}(f,1) &= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{1}{(z-1)^2(z^2+1)}\right]\\
&= \lim_{z\to 1}\frac{d}{dz}\left[\frac{1}{z^2+1}\right]\\
&= \lim_{z\to 1} -\frac{2z}{(z^2+1)^2}\\
&= -\frac{1}{2}
\end{aligned}$$

The integral can be evaluated as follows:

$$\int_{\gamma(0;2)}\frac{1}{(z-1)^2(z^2+1)}\,dz = 2\pi i\left(\frac{1}{4}+\frac{1}{4}-\frac{1}{2}\right) = 0$$

Now moving on to (2). We will use some work we did on Exercise 3.3 (4). The only difference is the contour only contains the singularities $z = \pi i/2$ and $z = -\pi i/2$. For any other $k \neq 0, -1$ the modulus will be too big $|z| > 2$. Since each pole is simple, we find

$$\operatorname{Res}\left(f, \pm\frac{\pi}{2}i\right) = \left.\frac{1}{\frac{d}{dz}(1+e^{2z})}\right|_{\pm\pi/2} = \frac{1}{2e^{\pm\pi i}} = -\frac{1}{2}$$

Since at each point the residue is $-\frac{1}{2}$, the integral can be evaluated as:

$$\int_{\gamma(0;2)}\frac{1}{1+e^{2z}}\,dz = 2\pi i(-1) = -2\pi i$$

Exercise 3.9 Use the calculus of residues to prove the following integrals.

1. $$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2ab(a+b)}, \qquad (a,b>0,\ a\neq b)$$
2. $$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx = \frac{\pi}{a}e^{-a}, \qquad (a>0)$$

■

Let us work on the first one. First notice that the integral is even since $f(-x) = f(x)$, which one can verify by noticing each x term is squared. Now we calcuate the improper integral from $-\infty$ to ∞. Here are the following steps we will take:

- First we will find the poles of complex $f(z)$ that lie in the upper half plane.
- We calculate the residues of these poles.
- We show that the integral over the semicircular arc vanishes as $R \to \infty$. Thankfully there are several tricks we can use for this. If the denominator is 2 or more degrees higher than the numerator, the integral will indeed vanish. If this doesn't work just use Jordans lemma or ML inequality.
- We apply residue theorem to the integral.

Consider the complex function:

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)} = \frac{1}{(z+ai)(z-ai)(z+bi)(z-bi)}$$

which has poles at $z = \pm ai$ and $z = \pm bi$. Since we are only concerned with those that lie in the upper half plane, we examine the poles $z = ai$ and $z = bi$. Since the poles are simple, we can caclulate their residues using the cover up method:

$$\text{Res}(f;ai) = \left.\frac{1}{(z+ai)(z+bi)(z-bi)}\right|_{z=ai} = \frac{1}{(ai+ai)(ai+bi)(ai-bi)} = \frac{1}{2ai(b^2-a^2)}$$
$$\text{Res}(f;bi) = \left.\frac{1}{(z+ai)(z-ai)(z+bi)}\right|_{z=bi} = \frac{1}{2bi(bi+ai)(bi-ai)} = \frac{1}{2bi(a^2-b^2)} = -\frac{1}{2bi(b^2-a^2)}$$

Now we set up the integral over our contour. Let C be the closed contour consisting of the real segment γ from $-R$ to R, and the upper half arc Γ. By residue theorem we have

$$\begin{aligned}
\int_C \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz &= 2\pi i \sum_{z_i} \text{Res}(f,z_i) \\
&= 2\pi i\left[\frac{1}{2ai(b^2-a^2)} - \frac{1}{2bi(b^2-a^2)}\right] \\
&= 2\pi i\left[\frac{b-a}{2i(b^2-a^2)ab}\right] \\
&= \frac{\pi(b-a)}{(b-a)(b+a)ab} \\
&= \frac{\pi}{ab(b+a)}
\end{aligned}$$

and we are able to break up the integral:

$$\frac{\pi}{ab(b+a)} = \oint_C \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz = \int_\gamma \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx + \int_\Gamma \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz$$
$$\int_{-R}^{R} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{ab(b+a)} - \int_\Gamma \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz$$
$$\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{ab(b+a)} - \lim_{R\to\infty}\int_\Gamma \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz$$
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{ab(b+a)} - \lim_{R\to\infty}\int_\Gamma \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz$$

Now the limit of the integral over Γ vanishes as the denominator is of degree 4 and the numerator is constant. Now since the integrand is even we can manipulate as follows:

$$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2ab(a+b)}$$

R The reason $a > 0$ and $b > 0$ was required was so that the poles were on the upper half of $\mathbb{C}$. The $a \neq b$ is not *required* but it makes the calculation easier, and if $a = b$ then we would have a completely different problem.

Let's move on to (2). We will proceed much like we did before save one exception. We will split the function like so:

$$f(z) = \frac{e^{iz}}{z^2+a^2} = \frac{\cos(z) + i\sin(z)}{z^2+a^2} = \frac{\cos(z)}{z^2+a^2} + \frac{i\sin(z)}{z^2+a^2}$$

This is a key observation, because the $\sin(z)$ integral, when taken from $-\infty$ to ∞ will converge to 0, since it is odd. Due to the way this works, we will have to apply this last. Let us first see that for

$$f(z) = \frac{e^{iz}}{z^2+a^2} = \frac{e^{iz}}{(z+ai)(z-ai)}$$

we have poles at $z = \pm ai$. Again, only ai is in the upper half plane. We calculate the residue at the simple pole ai using cover up method again:

$$\operatorname{Res}(f;ai) = \left.\frac{e^{iz}}{z+ai}\right|_{z=ai} = \frac{e^{i^2 a}}{ai+ai} = \frac{e^{-a}}{2ai}$$

We now set up the usual contour C as explained before. By residue theorem we will get

$$\begin{aligned} \oint_C \frac{e^{iz}}{z^2+a^2}\,dz &= 2\pi i\left[\frac{e^{-a}}{2ai}\right] \\ &= \frac{\pi}{a}e^{-a} \\ \int_\gamma \frac{e^{iz}}{x^2+a^2}\,dx + \int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz &= \frac{\pi}{a}e^{-a} \end{aligned}$$

And now we manipulate our contour integral as follows:

$$\begin{aligned} \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx + \int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz &= \frac{\pi}{a}e^{-a} \\ \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx &= \frac{\pi}{a}e^{-a} - \int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz \\ \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx &= \frac{\pi}{a}e^{-a} - \lim_{R\to\infty}\int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz \\ \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx &= \frac{\pi}{a}e^{-a} - \lim_{R\to\infty}\int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz \end{aligned}$$

We can now use Jordan's lemma. See that $|\frac{1}{z^2+a^2}| \leq \frac{1}{|z|^2+a^2} = \frac{1}{R^2+a^2}$ which goes to 0 as $R \to \infty$. Thus by Jordan's lemma the integral over Γ as $R \to \infty$ vanishes, and we are left with

$$\begin{aligned} \frac{\pi}{a}e^{-a} &= \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx \\ &= \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x^2+a^2}\,dx \\ &= \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx + \int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2}\,i\,dx \\ &= \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx + 0 \\ \therefore \qquad \frac{\pi}{a}e^{-a} &= \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx \end{aligned}$$

R The $a > 0$ was required so the pole in the upper half plane can easily be identified.