Part One

1. Prolouge

> It is a cliche, but true nevertheless, that math can be learned only by doing it.
>
> *Andrew Pressley*

1.1 Structure of these notes

These notes are a result of a 12 week semester course in multivariate calculus. There were 3 assignments, which were worth a total 25% of the course. They also provided preparation to the exams, each also worth 25%. Each chapter follows this rough structure:

- Walkthrough of the assignment problems
- Comments I have on the resources/content
- Notes for the reader, to summarize the content covered

Please note: These notes are *not* intended to be a resource for learning multivariate calculus. This merely serves as a *review for someone who has already taken multivariate calculus.* There are simply too many details left out from lectures/books/conversations I cannot possibly all include.

1.2 Actions speak louder than words

It is often said that mathematics is not a spectator sport. You don't learn math by watching others do it, you have to do it yourself. Thus, the only metric for progress is how well you can do the exercises presented (provided they are of good quality). The following provides not only the exercises, but the worked through solutions to the exercises.

2. Series and Curves

> The subject in which we never know what we are talking about, nor whether what we are saying is true.
>
> *Bernard Russell*

2.1 Assignment walkthrough

Exercise 2.1 Let $a_n = \ln n$, $b_n = n^p$, $c_n = a^n$ and $d_n = n!$, where a and p are constants. Show that for any numbers $p > 0$ and $a > 1$:

(a) $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$;

(b) $\lim_{n\to\infty} \frac{b_n}{c_n} = 0$;

(c) $\lim_{n\to\infty} \frac{c_n}{d_n} = 0$.

These relations along with the relation $n! \ll n^n$ establish the relation

$$\ln n \ll n^p \ll a^n \ll n! \ll n^n.$$

■

Starting the first one, we see that the function is of the form ∞/∞. This means we may apply L'Hopital's rule:

$$\lim_{n\to\infty} \frac{\ln n}{n^p} = \lim_{n\to\infty} \frac{\frac{d}{dx}\ln n}{\frac{d}{dx}n^p} = \lim_{n\to\infty} \frac{\frac{1}{n}}{np^{p-1}} = \lim_{n\to\infty} \frac{1}{pn^p} = 0$$

for any constant $p > 0$. Question (b) is similar, save that we will need to apply the rule k times:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^p}{a^n} &= \lim_{n\to\infty} \frac{pn^{p-1}}{a^n \ln a} \\
&= \lim_{n\to\infty} \frac{p(p-1)n^{p-2}}{a^n(\ln a)^2} \\
&= \lim_{n\to\infty} \frac{p(p-1)(p-2)n^{p-3}}{a^n(\ln a)^3} \\
&\vdots
\end{aligned}$$

And we keep doing this for k times as long as $k \geq p$ so that we will have

$$\lim_{n\to\infty}\frac{n^p}{a^n} = \lim_{n\to\infty}\frac{\overbrace{p(p-1)(p-2)\cdots(p-k+1)}^{\text{some constant } C}n^{p-k}}{a^n(\ln a)^k} = \lim_{n\to\infty}\frac{C}{a^n(\ln a)^k n^{k-p}}$$

Which is obviously 0 because all the terms in the denominator are positive. Now we solve (c):

Proof. The trick for this proof is to observe the maximum point of the sequence happens when $n = a$, after which the sequence is strictly decreasing. Thus one can see that

$$\left|\frac{a^n}{n!}\right| = \frac{a}{n}\cdot\overbrace{\frac{a}{n-1}\cdots\frac{a}{a}}^{\leq 1}\cdot\overbrace{\frac{a}{a-1}\cdots\frac{a}{1}}^{>1} \leq \frac{a}{n}\cdot\frac{a^a}{a!} = \frac{a^{a+1}}{a!}\cdot\frac{1}{n}$$

And using the squeeze theorem we can thus see that

$$0 < \left|\frac{a^n}{n!}\right| \leq \frac{C}{n}$$

for the defined constant $C = a^{n+1}/a!$, implies the sequence goes to 0. ■

Now each of these limits show that the denominator grows much faster than the numerator. Using this logic, (a) implies $\ln n \ll n^p$, (b) implies $n^p \ll a^n$ and (c) implies that $a^n \ll n!$. Given that $n! \ll n^n$ we can establish the following:

$$\ln n \ll n^p \ll a^n \ll n! \ll n^n$$

Exercise 2.2 Test the following series for convergence or divergence. Note any results or tests used. Test the following series for convergence or divergence. Note any results or tests used.

(a)
$$\sum_{n=1}^{\infty}\frac{n+\sqrt{n}+1}{n^3\ln(n+2)+n}.$$

(b)
$$\sum_{n=0}^{\infty}\frac{(2n)!}{2^{n^2}}.$$

(c)
$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n^n}-1\right)^n.$$

(d)
$$\sum_{n=1}^{\infty}\frac{\cos(n\pi)(n+\ln n)}{2n+1}.$$

■

The first question can easily be solved by comparison test. First note for any $n > 1$ that $n+\sqrt{n}+1 < 3n$. Thus we can bound the series above by

$$\frac{n+\sqrt{n}+1}{n^3\ln(n+2)+n} < \frac{3n}{n^3\ln(n+2)+n} < \frac{3n}{n^3\ln(n+2)} = \frac{3}{n^2\underbrace{\ln(n+2)}_{>1}} < \frac{3}{n^2}$$

Since $3\sum_{n=1}^{\infty} 1/n^2$ is a convergent series, by comparison test we know that (a) converges to a limit $L < \frac{\pi^2}{2}$. For (b) we will use the ratio test. Of course all terms are positive. We calculate the limit of the ratio:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n^2} \cdot (2n+2)!}{(2n)! \cdot 2^{n^2+2n+1}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{2^{2n+1}} = 0$$

Since it is less than one, we can conclude it **does converge**. For (c) we see in absolute terms that $|a_n| \to 0$ as $n \to \infty$ for

$$\left|\frac{1}{n^n} - 1\right| = \left(1 - \frac{1}{n^n}\right)^n$$

because $n^n \geq 1$ for all $n \geq 1$. Combined with the fact that the terms are monotonically decreasing we have it by alternating series test that it does converge. For (d) first see that $\cos(n\pi) = (-1)^n$. Thus

$$\lim_{n\to\infty} \frac{\cos(n\pi)(n+\ln n)}{2n+1} = \lim_{n\to\infty} (-1)^n \frac{(n+\ln n)}{2n+1} = \lim_{n\to\infty} (-1)^n \frac{(1+\frac{\ln n}{n})}{2+\frac{1}{n}} = \frac{1}{2} \lim_{n\to\infty} (-1)^n.$$

and thus (d) fails to converge by divergence test.

Exercise 2.3 For what values of k is the series

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln n (\ln(\ln n))^k}$$

absolutely convergent? For what values of $k \geq 0$ is the series conditionally convergent? ■

We will test the series for absolute convergence. This means we trying to see if $\sum_{n=1}^{\infty} |f(n)|$ converges. This means we can get rid of the $(-1)^n$ term. Now we may use the integral test (first see that the terms are clearly positive, decreasing, and continuous):

$$\int_3^{\infty} \frac{dx}{n \ln n (\ln(\ln n))^k} = \int_{\ln(\ln 3)}^{\infty} u^{-k} du$$

With the substitution $u = \ln(\ln n)$. Now we cant use the power rule for $k = -1$ so we shall do this by case:

- If $k = 1$:

$$\int_{\ln(\ln 3)}^{\infty} u^{-1} du = \ln u \Big]_{\ln(\ln 3)}^{\infty}$$

  which clearly diverges.
- If $k > 1$ then

$$\int_{\ln(\ln 3)}^{\infty} u^{-k} du = \frac{u^{-k+1}}{-k+1}\Big]_{\ln(\ln 3)}^{\infty} = 0 - \frac{(\ln(\ln 3))^{-k+1}}{-k+1} = \frac{(\ln(\ln 3))^{-k+1}}{k-1} = C$$

  which is convergent. Since the above relied on the fact that $u^{1-k} \to 0$ which only works when $k > 1$, we know it will not work when $k < 1$.

Thus, the series is absolutely convergent when $k \geq 1$. However, because the series in monotonically decreasing (for $n \geq 3$) and converges to 0 by alternating series test it converges for $0 \leq k \leq 1$.

Exercise 2.4 Find the radius of convergence and the interval of convergence for the following power series:

1. $$\sum_{n=1}^{\infty} \frac{2^n (x-2)^n}{(n+2)!};$$

2.
$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{\sqrt{n}}.$$

We start with ratio test
$$\left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\frac{(n+2)!}{2^n(x-2)^n}\right| = \left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\frac{(n+2)!}{2^n(x-2)^n}\right| = \left|\frac{2(x-2)}{(n+3)}\right|$$
$$|2x-4|\lim_{n\to\infty}\frac{1}{n+3} = 0$$
Thus, $R=\infty$, so the series (1) converges for all $x\in\mathbb{R}$. For (2) we apply the same technique:
$$\left|\frac{(-1)^{n+1}(x+1)^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{(-1)^n(x+1)^n}\right| = \frac{\sqrt{n}}{\sqrt{n+1}}|-(x+1)| = \frac{\sqrt{n}}{\sqrt{n+1}}|x+1|$$
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x+1|\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = |x+1|$$
We know the series (2) converges when $|x+1|<1$. Thus our interval of convergence becomes $(-2,0)$. We need only test the endpoints:
$$\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$
which is a diverging $p-$series for $p\le 1$. And at $x=0$ we have
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$
which does converge by alternating series test because $1/\sqrt{n}$ is monotonic decreasing, and approaches 0.

Exercise 2.5 Suppose that $a_n > 0$ for all n, and that $\frac{a_{n+1}}{a_n}\to L>0$. Show that the following series all have a radius of convergence $R=1/L$:
$$\sum a_n(x-x_0)^n,$$
$$\sum na_n(x-x_0)^n,$$
$$\sum \frac{a_n}{n}(x-x_0)^n.$$

For each equation, we will apply the ratio test, then use the supposition that $a_{n+1}/a_n = L$ as $n\to\infty$. This will put in the form of $|x-x_0|L<1$. This will prove that each of the series have a radius of convergence of $1/L$. Here we apply the ratio test to each equation:
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\frac{a_{n+1}}{a_n}$$
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)a_{n+1}(x-x_0)^{n+1}}{na_n(x-x_0)^n}\right| = |x-x_0|\frac{n+1}{n}\frac{a_{n+1}}{a_n}$$
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{na_{n+1}(x-x_0)^{n+1}}{(n+1)a_n(x-x_0)^n}\right| = |x-x_0|\frac{n}{n+1}\frac{a_{n+1}}{a_n}$$
Taking the limit of all of these as $n\to\infty$, they all equal $|x-x_0|L$. Because we want $|x-x_0|L<1$ we have a radius of converge of $1/L$.

Exercise 2.6 Let γ be the plane curve defined by the C^2 function

$$\mathbf{r}(t) = (x(t), y(t))$$

for $t \in [t_0, t_1]$.

1. Show that
$$\mathbf{T}'(t) = K|\mathbf{r}'(t)|\mathbf{N}_2$$
for
$$K = \frac{x''y' - x'y''}{|\mathbf{r}'(t)|^3} \quad , \quad \mathbf{N}_2 = \frac{(y', -x')}{|\mathbf{r}'(t)|}$$
2. Use the definition of curvature given in the Study Guide, the chain rule and part (a) to prove the curvature formula
$$\kappa = \frac{|x''y' - x'y''|}{|\mathbf{r}'|^3}$$

Let's make a few notes. By observation we can see that

$$K|\mathbf{r}'(t)|\mathbf{N}_2 = \frac{x''y' - x'y''}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|\frac{(y', -x')}{|\mathbf{r}'(t)|} = \frac{x''y' - x'y''}{|\mathbf{r}'(t)|^3}(y', -x')$$

We can verify this is equal to $\mathbf{T}'(t)$ by calculating:

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \\
&= \frac{(x', y')}{\sqrt{(x')^2 + (y')^2}} \\
&= \left(\frac{x'}{\sqrt{(x')^2 + (y')^2}}, \frac{y'}{\sqrt{(x')^2 + (y')^2}} \right)
\end{aligned}$$

Here we have $s(t) = \sqrt{(x')^2 + (y')^2}$. For the first component:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{x'}{s}\right) &= \frac{x''s - x'\frac{ds}{dt}}{s^2} \\
&= \frac{x''s - x'\left(\frac{x'x'' + y'y''}{s}\right)}{s^2} \\
&= \frac{x''s^2 - x'(x'x'' + y'y'')}{s^3} \\
&= \frac{x''(y')^2 - x'y'y''}{s^3}
\end{aligned}$$

For the second component:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{y'}{s}\right) &= \frac{y''s - y'\frac{ds}{dt}}{s^2} \\
&= \frac{y''s^2 - y'(x'x'' + y'y'')}{s^3} \\
&= \frac{y''(x')^2 - x'y'x''}{s^3}
\end{aligned}$$

Thus

$$\mathbf{T}'(t) = \left(\frac{x''(y')^2 - x'y'y''}{s^3}, \frac{y''(x')^2 - x'y'x''}{s^3} \right)$$

This can be rewritten as:

$$\mathbf{T}'(t) = \frac{x''y' - x'y''}{|\mathbf{r}'(t)|^3} \left(y', -x' \right)$$

Using the definition of curvature $\kappa(s) = |\mathbf{T}'(s)|$ and the results from part (1), we prove the curvature formula. From part (1), we have:

$$\mathbf{T}'(t) = K|\mathbf{r}'(t)|\mathbf{N}_2 = \frac{x''y' - x'y''}{|\mathbf{r}'(t)|^2}\mathbf{N}_2$$

where $\mathbf{N}_2$ is a unit vector. The arc length parameter s satisfies $ds/dt = |\mathbf{r}'(t)|$. By the chain rule:

$$\mathbf{T}'(s) = \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{\mathbf{T}'(t)}{|\mathbf{r}'(t)|}$$

Substituting $\mathbf{T}'(t)$:

$$\mathbf{T}'(s) = \frac{x''y' - x'y''}{|\mathbf{r}'(t)|^3}\mathbf{N}_2 = K\mathbf{N}_2$$

Taking the magnitude:

$$\kappa(s) = |\mathbf{T}'(s)| = \frac{|x''y' - x'y''|}{|\mathbf{r}'(t)|^3}|\mathbf{N}_2| = \frac{|x''y' - x'y''|}{|\mathbf{r}'(t)|^3}$$

since $|\mathbf{N}_2| = 1$.

Therefore, we obtain the curvature formula:

$$\kappa = \frac{|x''y' - x'y''|}{|\mathbf{r}'|^3}$$

Exercise 2.7 The space curve $\Gamma \subset \mathbb{R}^3$ is parameterized by

$$\mathbf{r}(t) := \left(t, t^2, \frac{2}{3}t^3 \right)$$

for $t \in [0,1]$.

1. Determine the unit tangent and normal vectors for Γ.
2. Determine the arc length function $s(t)$.

■

We first find the tangent vector at time t. We will first take the derivative;

$$\mathbf{r}'(t) = \left(1, 2t, 2t^2\right)$$

Notice because $|\mathbf{r}'(t)| \neq 0$ for all t it is a regular C^∞ curve. The length of the tangent vector is equal to $\sqrt{1+4t^2+4t^4} = \sqrt{(1+2t^2)^2} = 1+2t^2$. The unit tangent can be found by dividing the tangent by its length.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\left(1, 2t, 2t^2\right)}{2t^2+1} = \left(\frac{1}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{2t^2}{2t^2+1} \right)$$

Now to find the unit normal we follow the same idea:

$$\begin{aligned}
\mathbf{T}'(t) &= \left(-\frac{4t}{(2t^2+1)^2}, \frac{4t^2+2-8t^2}{(2t^2+1)^2}, \frac{8t^3+4t-8t^3}{(2t^2+1)^2}\right) \\
&= \left(-\frac{4t}{(2t^2+1)^2}, \frac{2-4t^2}{(2t^2+1)^2}, \frac{4t}{(2t^2+1)^2}\right) \\
|\mathbf{T}'(t)| &= \sqrt{\left(-\frac{4t}{(2t^2+1)^2}\right)^2 + \left(\frac{2-4t^2}{(2t^2+1)^2}\right)^2 + \left(\frac{4t}{(2t^2+1)^2}\right)^2} \\
&= \sqrt{\frac{16t^2+4-16t^2+16t^4+16t^2}{(2t^2+1)^4}} \\
&= \frac{\sqrt{16t^4+16t^2+4}}{(2t^2+1)^2} \\
&= \frac{\sqrt{4(4t^4+4t^2+1)}}{(2t^2+1)^2} = \frac{2(2t^2+1)}{(2t^2+1)^2} = \frac{2}{2t^2+1}
\end{aligned}$$

Finally, we get the unit normal vector:

$$\begin{aligned}
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \\
&= \left(-\frac{4t}{(2t^2+1)^2}, \frac{4t^2+2-8t^2}{(2t^2+1)^2}, \frac{8t^3+4t-8t^3}{(2t^2+1)^2}\right) \frac{2t^2+1}{2} \\
&= \left(-\frac{2t}{2t^2+1}, \frac{1-2t^2}{2t^2+1}, \frac{2t}{2t^2+1}\right)
\end{aligned}$$

We can get the arc length parametrization as follows:

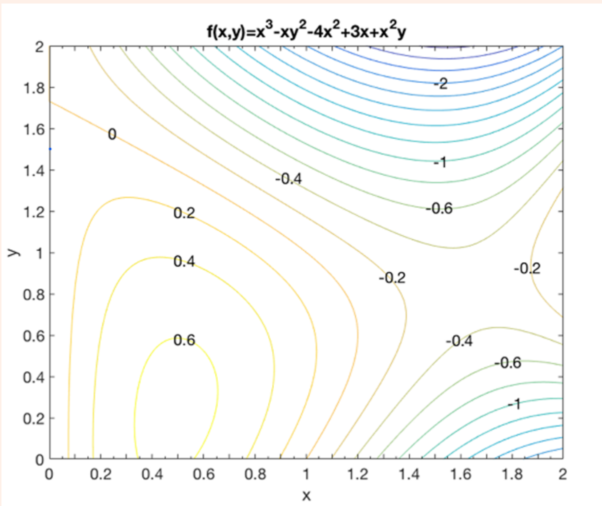
$$\begin{aligned}
s(t) := &\int_{t_0}^{t} |\mathbf{r}'(t)| \; dt \\
= &\int_0^t 2t^2+1 \, dt \\
= &\frac{2}{3}t^3 + t
\end{aligned}$$

3. Derivatives and Integration

> Young man, in mathematics you don't understand things. You just get used to them.
>
> *Von Neumann*

3.1 Assignment Problems

Exercise 3.1 We have the contour plot of the function $f(x,y) = x^3 - xy^2 - 4x^2 + 3x + x^2y$ for $(x,y) \in [0,2] \times [0,2]$:



1. Use the contour plot to determine whether f_x and f_y are greater than, less than, or equal to zero at the point $(1,1.5)$? What about at the point $(1.2,0.6)$? Explain your reasoning.
2. Calculate the actual values of f_x and f_y at those points.
3. Find the maximum and minimum values of the directional derivative $D_u f$ at $(1/2,1)$ as u varies.
4. In which directions u does the maximum and minimum occur?

5. Find the direction(s) u for which $D_u f(1/2, 1) = 0$.
6. Are your answers to (4) and (5) as to be expected from the contour plot?

We will answer these with corresponding numbers for ease of reading.

1. Looking at where $(1, 1.5)$ is located, we can see it is between the values of $-.6$ and -1. In both the x and y direction we are stepping into the -1 level, therefore $f_x, f_y < 0$. As for the point $(1.2, 0.6)$ we see it is located on the 0 level. In the x direction it is clear we are moving to the $-.2$ level, therefore $f_x < 0$. However, while it may seem like we are moving up to a negative contour, at that point the contour is effectively moving in the y direction with no change in value. Basically, as we take a small step in the y direction we are staying on the level. Therefore, since there is no change in value, we have $f_y = 0$.
2. We calculate:

$$f_x = 3x^2 - y^2 - 8x + 3 + 2xy$$
$$f_y = -2xy + x^2$$

Evaluating at the points we have $f_x(1, 1.5) = -1.25$ and $f_y(1, 1.5) = -2$. Similarly, $f_x(1.2, 0.6) = -1.2$ and $f_y(1.2, 0.6) = 0$.
3. We know the gradient is simply

$$\begin{aligned}\nabla f &= \langle f_x, f_y \rangle \\ &= \left\langle 3x^2 - y^2 - 8x + 3 + 2xy,\ x^2 - 2xy \right\rangle \\ \nabla f\left(\tfrac{1}{2}, 1\right) &= \left\langle -0.25,\ -0.75 \right\rangle \\ &= \left\langle -\frac{1}{4}, -\frac{3}{4} \right\rangle \\ \left\|\nabla f(\tfrac{1}{2}, 1)\right\| &= \sqrt{\left(-\tfrac{1}{4}\right)^2 + \left(-\tfrac{3}{4}\right)^2} = \frac{\sqrt{10}}{4}\end{aligned}$$

This means the maximum and minimum values of the directional derivative is $\sqrt{10}/4$ and $-\sqrt{10}/4$ respectively.
4. The directions for the maximum and minimum are

$$\left\langle -\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right\rangle \quad \text{and} \quad \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

5. To find when $\nabla f \cdot u = 0$ we will need to find two orthogonal us to This can be done from observation:

$$\left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle \quad \text{and} \quad \left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right\rangle$$

6. The directions for the maximum and minimum directional derivative as we found in (4) can be estimated to be $\approx \langle -.3, -1 \rangle$ and $\langle .3, 1 \rangle$ respectively. These are indeed perpendicular to the contour line at that point. Also notice that the vector $\langle -.3, -1 \rangle$ goes in the direction towards higher level contours confirming it is in the maximum direction. Likewise for the minimum, which points downwards. As for (5), we can see that those vectors are parallel (tangent) to the contour, which means it doesn't change, which means it's directional derivative in that direction (and the opposite) is 0.

Exercise 3.2 The voltage, V , (in volts) across a circuit is given by Ohms law: $V = IR$, where I is the current (in amps) owing through the circuit and R is the resistance (in ohms). If we place two circuits, with resistance R_1 and R_2, in parallel, then their combined resistance, R, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Suppose the current is 2 amps and is increasing at 10^{-2} amp/sec and R_1 is 3 ohms and increasing at 0.5 ohms/sec, while R_2 is 5 ohms and decreasing at 0.1 ohms/sec. Calculate the rate at which the voltage is changing. ■

We wish to find $\partial V/\partial t$. We have a formula for V so using chain rule we have:

$$\frac{\partial V}{\partial t} = I\frac{dR}{dt} + R\frac{dI}{dt}$$

Let's take stock of what we already know:

$$\frac{dI}{dt} = .01 \qquad I = 2 \qquad R = \frac{15}{8}$$

Therefore we need to find $\frac{dR}{dt}$. First differentiate both sides with respect to t:

$$\frac{d}{dt}\left(\frac{1}{R}\right) = \frac{d}{dt}\left(\frac{1}{R_1}\right) + \frac{d}{dt}\left(\frac{1}{R_2}\right)$$

Now using the chain rule we recall

$$\frac{d}{dt}\left(\frac{1}{f(t)}\right) = -f(t)^{-2} \times f'(t) = -\frac{f'(t)}{f(t)^2}$$

Applying this to our equation:

$$\begin{aligned} -\frac{R'}{R^2} &= -\frac{R_1'}{R_1^2} - \frac{R_2'}{R_2^2} \\ R' &= R^2\left(\frac{R_1'}{R_1^2} + \frac{R_2'}{R_2^2}\right) \end{aligned}$$

R This is going to get ugly.

We are now ready to substitute into the formula and get our answer:

$$\begin{aligned} \frac{\partial V}{\partial t} &= I\frac{dR}{dt} + R\frac{dI}{dt} \\ &= IR^2\left(\frac{R_1'}{R_1^2} + \frac{R_2'}{R_2^2}\right) + R\frac{dI}{dt} \\ &= 2\left(\frac{15}{8}\right)^2\left(\frac{.5}{9} - \frac{.1}{25}\right) + \frac{15}{8}\frac{1}{100} \\ &= \frac{13225}{32}\left(\frac{58}{1125}\right) + \frac{15}{800} \\ &= \frac{767050}{36000} + \frac{15}{800} \\ &= \frac{61}{160} \\ &= .38125 \end{aligned}$$

R That wasn't so bad...

Exercise 3.3 On a certain mountain, the elevation z above a point (x,y) in the $xy-$plane at sea level is given by

$$z = 2500 - \frac{x^2}{10} - \frac{y^2}{5}$$

meters. The positive $x-$axis points east, and the positive $y-$axis points north. Suppose the climber is at the point $(15,-10,2457.5)$.

1. If the climber moves due west, will the climber begin to ascend or descend? At what rate?
2. If the climber moves southeast, will the climber begin to ascend or descend? At what rate?
3. In what direction should the climber move to travel a level path (describe the direction by giving a vector in that direction).

■

First compute the gradient

$$\begin{aligned}\nabla z &= \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle \\ &= \left\langle -\frac{x}{5}, -\frac{2y}{5} \right\rangle\end{aligned}$$

Which, when evaluated at the point of interest, gives us the vector $\langle -3,4 \rangle$. To compute the directional derivative in the west direction vector, $\langle -1,0 \rangle$ we simply take dot product:

$$D_{\langle -1,0 \rangle} z = \nabla z \cdot \langle -1,0 \rangle = \langle -3,4 \rangle \cdot \langle -1,0 \rangle = 3$$

This means for every meter the climber walks towards the west, the climber will ascend at a rate of 3 meters. To do (2) we use the normalized south-east vector $\frac{1}{\sqrt{2}}\langle 1,-1 \rangle$ and calculate

$$D_{\frac{1}{\sqrt{2}}\langle 1,-1 \rangle} z = \nabla z \cdot \frac{1}{\sqrt{2}}\langle 1,-1 \rangle = \langle -3,4 \rangle \cdot \langle 1/\sqrt{2}, -1\sqrt{2} \rangle = -\frac{7}{\sqrt{2}}$$

So as the climber moves one meter south east, he is descending down the mountain at ≈ 4.95 meters. For (3) we want the directional derivative to be zero. From observation, the vector $v = \langle 4,3 \rangle$ will work. Clearly, he could also go in the opposite direction $v = \langle -4,-3 \rangle$.

Exercise 3.4 Let

$$f(x,y) = x^2 + y^2 - \frac{1}{2}x^2 y$$

Find all the stationary points of f and determine whether each is a local maximum, a local minimum or a saddle point. ■

First we shall compute the partial derivatives:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x - xy \\ \frac{\partial f}{\partial y} &= 2y - \frac{1}{2}x^2\end{aligned}$$

And we set $\partial_x = 0$, $\partial_y = 0$ and solve the system. Solving the first we have $y = 2$ or $x = 0$. If $x = 0$ then the second equation reduces to $y = 0$. However, if $y = 2$ the second equation implies

$y = \pm\sqrt{8} = \pm 2\sqrt{2}$. These give the three critical points: $(0,0), (\pm 2\sqrt{2}, 2)$. The Hessian of f and it's determinant is given by

$$\nabla^2 f = \begin{bmatrix} 2-y & -x \\ -x & 2 \end{bmatrix}$$
$$\det(\nabla^2 f) = 4 - 2y - x^2$$

- Plugging in the first point $(0,0)$ gives the value 4. This classifies the point as a local minimum.
- Using the second point $(2\sqrt{2}, 2)$ yields the negative value -8, thus we know it's a saddle.
- Using the third point $(-2\sqrt{2}, 2)$ gives us the value -8 also. Thus it is also a saddle point.

Exercise 3.5 In this question we find the minimum and maximum values of

$$f(x,y) = x^2 - xy + y^2$$

inside the quarter circle given by $x^2 + y^2 \leq 1$ with $x, y \geq 0$. Do this following this outline:

1. Find the critical points that are strictly within the boundary.
2. Find the critical points, maxima and minima on each of the three boundary segments. Use the method of Lagrange multipliers for the curved segment.
3. Collate and compare the critical and extreme points found in (1) and (2) and hence identify the minimum and maximum values and their locations.

Plainly from observation we can see that $(0,0)$ is a critical point, but it is not "strictly" within the boundary, rather on it.

R I'm not sure if the question could perhaps mean something else. But for full marks I will show my working. Note $\partial_x = 2x - y$ and $\partial_y = 2y - x$. Setting $\partial_x = 0, \partial_y = 0$ and solving yields $(0,0)$ to be the critical point with a value of $f(0,0) = 0$.

As for (2) we will first work on the segment of the $y-$axis, where the function becomes $f(0,y) = y^2$. The critical points on the segment lie where $f'(0,y) = 0$, i.e. where $2y = 0$. This is at the point 0, which corresponds to the point $(0,0)$. Checking the endpoints $y = 0$ and $y = 1$ on the function f gives the maximum and minimum points $(0,1)$ and $(0,0)$ respectively.

We do likewise for the $x-$axis segment where the function becomes $f(x,0) = x^2$. Following the mirror procedure, we find the critical point to be $(0,0)$, which is also the minimum. We find the maximum to be $(1,0)$ with a value of 1.

Now we will do the curved path using the Lagrange multiplier. Here our constraint function is $g(x,y) = x^2 + y^2$, subject to $g(x,y) = 1$. Using our trusty equation:

$$\nabla f = \lambda \nabla g$$
$$\langle 2x - y, 2y - x \rangle = \lambda \langle 2x, 2y \rangle$$

So we have $2x - y = \lambda 2x$ which means that $\lambda = 1 - y/2x$. So we have 3 equations:

$$\lambda = 1 - \frac{y}{2x}$$
$$2y - x = \lambda 2y$$
$$x^2 + y^2 = 1$$

Note the first only holds if $x \neq 0$, which we have already covered. Substituting the first into the second equation we have $x^2 = y^2$. Clearly for this and the last equation to hold, then $x^2 = y^2 = 1/2$ Therefore $x, y = 1/\sqrt{2}$. We disregard the negative square roots due to our restraints on $x, y \geq 0$. This means our critical point on the boundary is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. To summarize, we will finish with (3).

- For (1), there is no critical point of f *strictly* within the boundary. However, there does exist a critical point $(0,0)$ with a value of 0 on the border.
- From (2), the points we found of interest were $(0,0), (0,1)$ on the $y-$axis. Also $(0,0), (1,0)$ on the $x-$axis, and lastly $(1/\sqrt{2}, 1/\sqrt{2})$ on the curved segment.
- Evaluating each of these, show that the points $(0,1)$ and $(1,0)$ are maximums with values of 1. If we want to include the point $(0,0)$ then we know it is a minimum with a value of 0.
- The critical value on the curved segment has a value of $1/2$, but is unimportant since it is not a maximum $= 1$, nor is it a minimum $= 0$.

Exercise 3.6 Let $I = \int\int_R (x+y)\,dA$ where R is the region bounded by the curve $y = x^2$ and the line $y = 1$.

1. Calculate I by integrating first with respect to y and then with respect to x.
2. Calculate I by integrating first with respect to x and then with respect to y.

■

Lets first examine the "free" variable, then fix it and see how the other variable behaves. For (1) we see that x ranges from -1 to 1. For each x, the y ranges from x^2 to 1. So calculating:

$$\begin{aligned}\int_{-1}^{1}\int_{x^2}^{1}(x+y)\,dy\,dx &= \int_{-1}^{1}\left(x+\frac{1}{2}-x^3-\frac{x^4}{2}\right)dx \\ &= \left[\left(\frac{x^2}{2}+\frac{x}{2}-\frac{x^4}{4}-\frac{x^5}{10}\right)\right]_{-1}^{1} \\ &= \frac{4}{5}\end{aligned}$$

Now let's look at things the other way around. We see that y runs from 0 to 1. For each fixed y, x is bounded between $-\sqrt{y}$ to $\sqrt{y}$. So calculating:

$$\begin{aligned}\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}(x+y)\,dx\,dy &= \int_{0}^{1} 2y^{\frac{3}{2}}\,dy \\ &= \left[\frac{4y^{\frac{5}{2}}}{5}\right]_{0}^{1} = \frac{4}{5}\end{aligned}$$

Exercise 3.7 Evaluate the iterated integral

$$\int_{0}^{1}\int_{x/2}^{1/2} e^{y^2}\,dy\,dx$$

by reversing the order of integration. ■

From the evaluations on the integral symbols we sketch the region we integrate over to be:

$$R = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1] \quad \text{and} \quad y \in [x/2, 1/2]\}$$

Reversing the order we let y range from 0 to $1/2$, and for each fixed y, we let x range from 0 to $2y$.

So swapping the order of integration means we only have to change the signs:

$$\int_0^1 \int_{x/2}^{1/2} e^{y^2}\, dy\, dx = \int_0^{1/2} \int_0^{2y} e^{y^2}\, dx\, dy$$
$$= \int_0^{1/2} 2y\, e^{y^2}\, dy$$

We can use substitution. Let $u = y^2$. So $du = 2y\, dy$. When $y = 0$ then $u = 0$, and when $y = 1/2$, then $u = 1/4$. So we have

$$\int_0^{1/2} 2y\, e^{y^2}\, dy = \int_0^{1/4} e^u du$$
$$= e^{\frac{1}{4}} - 1$$

Exercise 3.8 Let $\Omega \subset \mathbb{R}^3$ be the surface defined by $z = 1 - x^2 - y^2$.

1. By putting $z = 0$, find the curve of intersection of the surface Ω with the $xy-$plane.
2. Use polar coordinates to find the volume of the solid under the surface Ω and above the $xy-$plane.

■

For (1), by letting $z = 0$ we are left with ther level set of the circle defined by

$$x^2 + y^2 = 1$$

Since this is already in the xy plane we can leave it at that. The curve of intersection can be nicely represented in polar coordinates (if one wishes) by

$$(\cos t, \sin t) \quad \text{for} \quad t \in [0, 2\pi)$$

Lastly for (2) we are going to integrate $1 - x^2 - y^2$ by the region bounded by what we found in (1). Here, since it is a disk, we simply restrict the polar coordinates $(r, \theta) \in [0,1] \times [0, 2\pi)$. Let's simply calculate:

$$\int\int_{x^2+y^2 \le 1} (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2\cos^2\theta - r^2\sin^2\theta)\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 (1 - r^2) r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta$$
$$= \frac{1}{4} \int_0^{2\pi} d\theta$$
$$= \frac{\pi}{2}$$

R This problem was already covered in lesson 8.3 as example 2.

3.2 Comments

Exercise 2.1 makes use of a contour plot. This actually provides great intuition for understanding the gradient, and related topics such as the Lagrange multipliers. Exercise 2.2 makes use of the multivariable chain rule, this is very important for future work. In exercise 2.4 we can use the determinant of the Hessian to determine if a point is a saddle, local minimum, or local maximum. This is related to finding the Gauss curvature of a hypersurface in terms of the first fundamental form.

3.3 Notes and concepts to remember

Let's walk through a couple things.

3.3.1 Directional Derivatives

Say we are given a function of two variables, $f(x,y)$. We have explored the meaning of the partial derivatives. Here, $\frac{\partial f}{\partial x}$ is the rate of change of the function f in the *direction* of x. Similarly for $\frac{\partial f}{\partial y}$. We can generalize this notion to *any direction*.

Definition 3.3.1 Give a **unit** direction $\mathbf{u} = \langle u_1, u_2\rangle$, the **directional derivative** of $f(x,y)$ along the direction $\mathbf{u}$ at the point (x,y) is defined as

$$D_{\mathbf{u}} f(x,y) := \frac{d}{dt} f(x+tu_1, y+tu_2)\Big|_{t=0}$$

Notice if we let $\mathbf{u} = \langle 1,0\rangle$ which is the unit vector in the x direction we get

$$D_{\langle 1,0\rangle} f(x,y) = \frac{d}{dt} f(x+t,y)\Big|_{t=0} = \lim_{t\to 0} \frac{f(x+t,y)-f(x,y)}{t} = \frac{\partial f}{\partial x}(x,y)$$

■ **Example 3.1** We will compute the directional derivative of $f(x,y) = xy$ along the unit direction $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5}\rangle$, at the point $(2,3)$. Compute:

$$\begin{aligned} D_{\langle \frac{3}{5},\frac{4}{5}\rangle} f(x,y) &= \frac{d}{dt} f\left(2+\frac{3}{5}t, 3+\frac{4}{5}t\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(6+\frac{8}{5}t+\frac{9}{5}t+\frac{12}{5}t^2\right)\Big|_{t=0} \\ &= \left[\frac{17}{5}+\frac{24}{5}t\right]_{t=0} \\ &= \frac{17}{5} \end{aligned}$$

■

As one can see, it is a little tedious to do this computation every time. We have the following theorem to help us:

Theorem 3.3.1 Given a function $f(x,y)$ that is C^1 on its domain, then the directional derivative of f in the **unit direction** $\mathbf{u} = \langle u_1, u_2\rangle$ is given by

$$D_{\mathbf{u}} f(x,y) = \nabla f \cdot \mathbf{u}$$

The above is quite amazing, and its proof can be found in any standard textbook. We present one last theorem related to problem 1.

Theorem 3.3.2 Let $f(x,y)$ be a two-variable function which is C^1 on its domain, and (a,b) be a point on the level curve $f(x,y) = c$. Then the gradient vector $\nabla f(a,b)$ is orthogonal to the level curve $f(x,y) = c$ at the point (a,b).

3.3.2 Tangent Planes

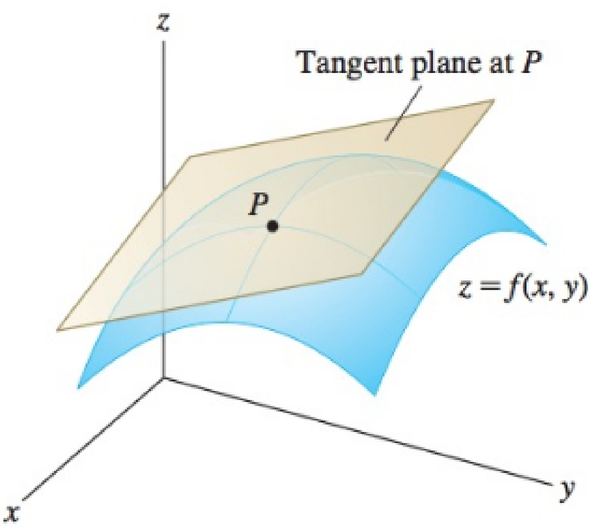
While not taught in the course, it is quite important to learn these topics. We will utilize the tools developed in the previous sub-section. For a function $y = f(x)$ at some particular point $(x_0, f(x_0))$ has slope $f'(x_0)$. Thus, the equation of the tangent line can be given by

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Now for a higher dimensional analogue, we have a multivariate function $z = f(x,y)$ and a point $f(x_0,y_0)$ for which we want to find a *tangent plane*. The following theorem is usually presented:

Theorem 3.3.3 Given a function $f(x,y)$ which is C^1 on its domain. The equation of the tangent plane for the graph $z = f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0))$ is given by:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y - y_0)$$



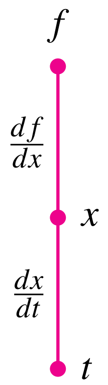
But why is this true? The best way is to illustrate with some examples (write this part soon).

3.3.3 Multivariable Differentiation

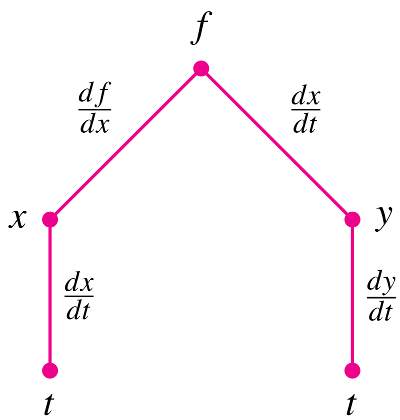
Recall to take the derivative of a function $f(x)$ you simply use some rule to find it. If a function is more complicated however, such as $f(x) = \sin(3t^2)$ you need to use the *chain rule*. This is given by:

$$\frac{d}{dx}f(x) = \frac{df}{dx}\frac{dx}{dt}$$

so in the previous example we have $x = 3t^2$, giving us $df/dx = \cos(3t^2)$ and $dx/dt = 6x$, which means $df/dt = 6x\cos(3x^2)$. This can be represented in a tree graph pictured below:



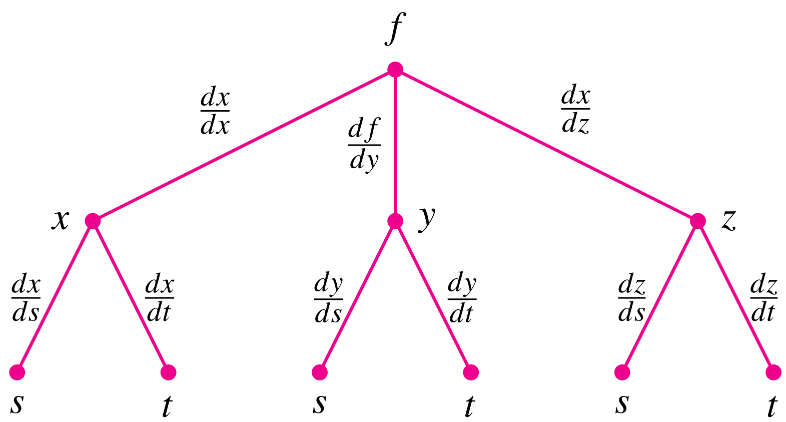
Which is great and all, but say we have a function $f(x,y)$ in which x depends on t and y also depends on t. Then we have the following tree:



Then to find the derivative of f with respect to t is simple. First find all paths from f to t. On each path multiply the derivatives you have to "cross". For example, on the path $f \to x \to t$ you want to get $\frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$. The derivative is then the sum of all these expressions from each path:

$$\frac{d}{dt} f(x,y) = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Obviously this works for more complicated functions as well. Take a function f that is dependent on 3 variables x, y and z. Each of these in turn are dependent on s and t. Then our tree would look like:



Then we get both

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

■ **Example 3.2** Let $f(x,y,z) = 3x^2 + y - \ln z$ where $\langle x,y,z\rangle = \langle s^2+t, st^3, \frac{s}{t}\rangle$. Find $\frac{\partial f}{\partial s}$. Here we use the chain rule:

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \\ &= 6x \cdot 2s + 1 \cdot t^3 - \frac{1}{z} \cdot \frac{1}{t}\end{aligned}$$

but we still have the x and z terms, so we will substitute:

$$\begin{aligned}\frac{\partial f}{\partial s} &= 6x \cdot 2s + 1 \cdot t^3 - \frac{1}{z} \cdot \frac{1}{t} \\ &= 12s(s^2+t) + t^3 - \frac{t}{s}\frac{1}{t} \\ &= 12s(s^2+t) + t^3 - \frac{1}{s}\end{aligned}$$

■

3.3.4 Application to Implicit Differentiation

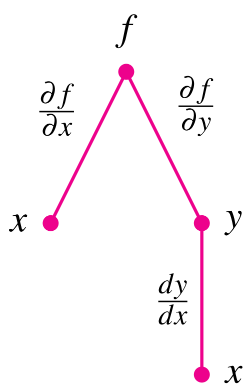
Say we have the *implicit equation* $\cos^2(y) + 3x^2 + y^4 = 4$. Here y is a function of x. Let's say we wanted to find y'. In that case, we differentiate both sides with respect to x to get:

$$-2\cos(y)\sin(y)\frac{dy}{dx} + 6x + 4y^3\frac{dy}{dx} = 0$$

Notice the above is *very complicated*. We then continue to solve for $\frac{dy}{dx}$ to get:

$$\begin{aligned}\left(4y^3 - 2\cos(y)\sin(y)\right)\frac{dy}{dx} &= -6x \\ \frac{dy}{dx} &= -\frac{6x}{4y^3 - 2\cos(y)\sin(y)}\end{aligned}$$

We can define $f(x,y) = \cos^2(y) + 3x^2 + y^4$, and we can then rewrite our implicit equation as $f(x,y) = 4$. Also, since f is dependent on x and y, and in turn y is dependent on x we have the following tree:



Which gives us the equation:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

but we are not done yet because we want $\frac{dy}{dx}$. Recall that $f(x,y) = 4$ which means $\frac{df}{dx} = 0$. Thus we can solve:

$$\begin{aligned}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} &= 0 \\ f_y \frac{dy}{dx} &= -f_x \\ \frac{dy}{dx} &= -\frac{f_x}{f_y}\end{aligned}$$

which aligns with our previous result. We can actually state this as a theorem:

Theorem 3.3.4 Given a function $y(x)$, and an implicit equation of the form $f(x,y) = c$, then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

4. Vector Fields

> If you're going through hell, keep going.
>
> *Winston Churchill*

4.1 Assignment Problems

These are the final exercise problems and notes based on my submission of assignment 3 of multivariate calculus. We are finishing the course with some more advanced topics. We will first go through the assignment problems, then summarize the concepts learned.

Exercise 4.1 The outside shell of a buried hollow storage tank is defined by the following equation:

$$x^2 + y^2 + (z+1)^4 = 1$$

1. Use cylindrical polar coordinates to calculate the volume of the tank.
2. The tank is now filled with a mixture of sand particles of varying iron content. These are placed so that the iron density at any point within the tank is given by

$$\rho(x,y,z) = \frac{x^2(2-z)}{x^2+y^2}$$

Use this to find the total iron content in the tank by using the formula

$$\iiint_{\text{Tank}} \rho \, dV.$$

■

We will use the cartesian to polar transformation with relations:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Which makes our equation:

$$\begin{aligned} x^2+y^2+(z+1)^4 &\leq 1 \\ r^2+(z+1)^4 &\leq 1 \\ r^2+u^4 &\leq 1 \end{aligned}$$

Above we substituted $u = z+1$ so we can more easily calculate our bounds. We shall now decide what order to do our bounds. If we choose u first then it would be above by $\sqrt[4]{1-r^2}$ which would be rather ugly. Bounding r above by $r \leq \sqrt{1-u^4}$ looks more appealing. By our definition of cylindrical coordinates r is bounded below by $r = 0$. It would make sense to then bound u. I'm not sure if one could call this a trick, but since $\sqrt{1-u^4} \geq r \geq 0$, we have $1 \geq u^4 \geq 0$, which implies $|u| \leq 1$. This means we have the bounds $-1 \leq u \leq 1$. Lastly, θ is bounded by definition, from $\theta \in [0, 2\pi]$.

R Note that $-1 \leq u \leq 1$ implies $-2 \leq z \leq 0$, this will be used in our next calculation.

Since our order is $dV = r\,dr\,du\,d\theta$, we calculate:

$$\int_0^{2\pi}\int_{-1}^{1}\int_0^{\sqrt{1-u^4}} r\,dr\,du\,d\theta$$

Solving the inner integral:

$$\int_0^{\sqrt{1-u^4}} r\,dr = \left[\frac{r^2}{2}\right]_0^{\sqrt{1-u^4}} = \frac{1-u^4}{2} = \frac{1}{2} - \frac{u^4}{2}$$

Solving the next part of the integral:

$$\begin{aligned} \int_{-1}^{1} \frac{1}{2} - \frac{u^4}{2}\,du &= \left[\frac{u}{2} - \frac{u^5}{10}\right]_{-1}^{1} \\ &= \left[\frac{5u-u^5}{10}\right]_{-1}^{1} \\ &= \frac{4}{10} + \frac{4}{10} = \frac{8}{10} \end{aligned}$$

Which leaves us with our last integral:

$$\int_0^{2\pi} \frac{8}{10}\,d\theta = \left[\frac{8\theta}{10}\right]_0^{2\pi} = \frac{8\pi}{5}$$

For (2) we pretty much do the same. Start by changing basis of ρ:

$$\rho = \frac{r^2\cos^2\theta(3-u)}{r^2} = \cos^2\theta(3-u)$$

So we need to calculate

$$\iiint_{\text{Tank}} \rho\,dV = \int_0^{2\pi}\int_{-1}^{1}\int_0^{\sqrt{1-u^4}} \cos^2\theta(3-u)\,r\,dr\,du\,d\theta$$

The outer integral can pull out the $\cos^2\theta$ so we simplify to

$$\int_0^{2\pi}\cos^2\theta\,d\theta = \pi$$

$$\iiint_{\text{Tank}}\rho\,dV = \left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)\int_{-1}^{1}\int_0^{\sqrt{1-u^4}}\theta(3-u)\,r\,dr\,du$$

$$= \pi\int_{-1}^{1}\int_0^{\sqrt{1-u^4}}(3-u)\,r\,dr\,du$$

Lets integrate the inner integral:

$$\int_0^{\sqrt{1-u^4}} r\,dr = \left[\frac{r^2}{2}\right]_0^{\sqrt{1-u^4}} = \frac{1-u^4}{2}$$

Thus our original integral reduces to

$$\begin{aligned}\iiint_{\text{Tank}}\rho\,dV &= \pi\int_{-1}^{1}\int_0^{\sqrt{1-u^4}}(3-u)\,r\,dr\,du\\ &= \frac{\pi}{2}\int_{-1}^{1}(3-u)(1-u^4)\,du\\ &= \frac{\pi}{2}\left[3u-\frac{3}{5}u^5-\frac{1}{2}u^2-\frac{1}{6}u^6\right]_{-1}^{1}\\ &= \frac{\pi}{2}\frac{24}{5}\\ &= \frac{12\pi}{5}\end{aligned}$$

Exercise 4.2 Use the change of variables $u = x+y$ and $v = x-y$ to calculate

$$\iint_R (x^2-y^2)\sin(x+y)\,dA$$

where R is the region in the $xy-$plane enclosed by the line $y = x+\pi$ and $y = \pi - x$, and the $x-$axis. ■

Lets get our inverses:

$$x = \frac{u+v}{2} \quad , \quad y = \frac{u-v}{2}$$

This means that $y = x+\pi$ implies that $v = -\pi$, $y = 0$ implies that $u = v$, and lastly $y = \pi - x$ implies $u = \pi$. Our Jacobian is calculated to be $J(u,v) = -1/2$. Finally we may solve the integral:

$$\begin{aligned}\iint_R (x^2-y^2)\sin(x+y)\,dA &= \iint_R uv\sin(u)\,|J(u,v)|\,dA\\ &= \frac{1}{2}\iint_R uv\sin(u)\,dA\\ &= \frac{1}{2}\int_{-\pi}^{\pi}\int_{-\pi}^{u} uv\sin(u)\,dv\,du\\ &= \frac{1}{2}\int_{-\pi}^{\pi} u\sin(u)\left(\int_{-\pi}^{u} v\,dv\right)du\\ &= \frac{1}{4}\int_{-\pi}^{\pi}\sin(u)(u^3-u\pi^2)\,du\end{aligned}$$

To solve this we use integration by parts using our DI method:

Sign	D	I
$+$	$u^3 - u\pi^2$	$\sin(u)$
$-$	$3u^2 - \pi^2$	$-\cos(u)$
$+$	$6u$	$-\sin(u)$
$-$	6	$\cos(u)$
$+$	0	$-\sin(u)$

This reduces our final integral to

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(u)(u^3 - u\pi^2)\,du &= \left[-\cos(u)(u^3 - u\pi^2) + \sin(u)(3u^2 - \pi^2) + 6u\cos(u) - 6\sin(u)\right]_{-\pi}^{\pi} \\ &= -6\pi - (-6\pi) \\ &= -12\pi\end{aligned}$$

Which means our final answer is

$$\frac{1}{2}\int_{-\pi}^{\pi}\int_{-\pi}^{u} uv\sin(u)\,dv\,du = -\frac{12\pi}{4} = -3\pi$$

Exercise 4.3 Calculate the line integral $\int_C F \cdot d\mathbf{r}$ where

$$F(x,y,z) := xyz\,\mathbf{i} + yz\,\mathbf{j} + z\,\mathbf{k}$$

and C is the parameterised curve

$$\mathbf{r}(t) := e^t\,\mathbf{i} + e^{2t}\,\mathbf{j} + e^{4t}\,\mathbf{k}, \quad 0 \le t \le 1$$

■

This is just a boring calculation. Notice that:

$$\begin{aligned}\mathbf{r}'(t) &= \left\langle e^t, 2e^{2t}, 4e^{4t}\right\rangle \\ F(t) &= \left\langle e^t e^{2t} e^{4t}, e^{2t} e^{4t}, e^{4t}\right\rangle \\ &= \left\langle e^{7t}, e^{6t}, e^{4t}\right\rangle \\ F \cdot \mathbf{r}'(t) &= \langle xyz, yz, z\rangle \cdot \left\langle e^t, 2e^{2t}, 4e^{4t}\right\rangle \\ &= \left\langle e^{7t}, e^{6t}, e^{4t}\right\rangle \cdot \left\langle e^t, 2e^{2t}, 4e^{4t}\right\rangle \\ &= e^{8t} + 2e^{8t} + 4e^{8t} \\ &= 7e^{8t} \\ \int_0^1 7e^{8t}\,dt &= 7\left[\frac{e^{8t}}{8}\right]_0^1 \\ &= \frac{7}{8}\left(e^8 - 1\right)\end{aligned}$$

Exercise 4.4 Use Green's theorem to evaluate

$$\oint_C (x^2y + y^3)\,dx + (2xy + \sin(y))\,dy$$

where $C = \partial D$ is the positively oriented curve that encloses the region D bounded by the parabola $y = 1 - x^2$, the x-axis and the y-axis in the 1st quadrant. ■

Let's first find the bounds. Notice that x enters D at 0 and out at $\sqrt{1-y}$. Then for each slice, y runs from 0 to 1. That done, we calculate:

$$\begin{aligned}
\int_C \overbrace{(x^2y+y^3)}^{P}\,dx + \overbrace{(2xy+\sin(y))}^{Q}\,dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \\
&= \iint_D 2y - x^2 - 3y^2\,dx\,dy \\
&= \int_0^1 \int_0^{\sqrt{1-y}} 2y - x^2 - 3y^2\,dx\,dy \\
&= \int_0^1 \left[2xy - \frac{x^3}{3} - 3xy^2\right]_0^{\sqrt{1-y}} dy \\
&= \int_0^1 2y\sqrt{1-y} - \frac{(1-y)^{\frac{3}{2}}}{3} - 3y^2\sqrt{1-y}\,dy
\end{aligned}$$

Now this is a rather horrible creature to integrate, so let us slay it by first decapitating its 3 heads, and *integrating* each one without mercy. For first:

$$\int_0^1 2y\sqrt{1-y}\,dy$$

we will use the substitution $u = 1 - y$, with $dy = -du$. Notice as $y = 0$ then $u = 1$, and as $y = 1$ then $u = 0$. Thus:

$$\begin{aligned}
\int_0^1 2y\sqrt{1-y}\,dy &= 2\int_1^0 (1-u)\sqrt{u}\,(-du) \\
&= 2\int_0^1 u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du \\
&= 2\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1 \\
&= 2\left[\frac{10}{15} - \frac{6}{15}\right] \\
&= \frac{8}{15}
\end{aligned}$$

Two more to go. Here we use the same sub:

$$\begin{aligned}
-\frac{1}{3}\int_0^1 (1-y)^{\frac{3}{2}}\,dy &= -\frac{1}{3}\int_0^1 u^{\frac{3}{2}}\,du \\
&= -\frac{1}{3}\left[\frac{2}{5}u^{\frac{5}{2}}\right]_0^1 \\
&= -\frac{2}{15}
\end{aligned}$$

And the final boss:

$$\begin{aligned}
-3\int_0^1 y^2\sqrt{1-y}\,dy &= -3\int_0^1 (1-u)^2\sqrt{u}\,du \\
&= -3\int_0^1 (1-2u+u^2)u^{\frac{1}{2}}\,du \\
&= -3\int_0^1 u^{\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{5}{2}}\,du \\
&= -3\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{7}u^{\frac{7}{2}}\right]_0^1 \\
&= -3\left[\frac{70}{105} - \frac{84}{105} + \frac{30}{105}\right] \\
&= \frac{16}{35}
\end{aligned}$$

We have slain the beast. Our final solution (let us pray it is the right answer):

$$\int_C (x^2y+y^3)\,dx + (2xy+\sin(y))\,dy = \frac{8}{15} - \frac{2}{15} + \frac{16}{35} = -\frac{2}{35}$$

Exercise 4.5 Find the curl and divergence of the following vector field:

$$F(x,y,z) = xy\,\mathbf{i} + xz\,\mathbf{j} + xy^2z\,\mathbf{k}$$

■

For the curl simply calculate:

$$\begin{aligned}
\operatorname{curl}(F) = \nabla\times F = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & xy^2z \end{bmatrix} \\
= \left\langle (2xyz - x), -(y^2z-0), (z-x)\right\rangle \\
= \left\langle 2xyz - x, -y^2z, z-x\right\rangle
\end{aligned}$$

The divergence can be calculated by :

$$\operatorname{div}(F) = \nabla\cdot F = y + xy^2$$

Exercise 4.6 Find

$$\iint_S (x-y+z^2)\,dS$$

where S is the triangle with parametric equations:

$$\mathbf{r}(s,t) := \langle s+2t, 2s+t, 3s\rangle$$

for $0\leq s\leq 1-t$ and $0\leq t\leq 1$. ■

Let's first substitute to find:

$$x - y + z^2 = s + 2t - 2s - t + 9s^2 = t - s + 9s^2$$

And its area form can be found as such:

$$
\begin{aligned}
dS &= \left\| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right\| \\
&= \| \langle 1,2,3 \rangle \times \langle 2,1,0 \rangle \| \\
&= \| \langle -3,6,-3 \rangle \| \\
&= \sqrt{9+36+9} = 3\sqrt{6}
\end{aligned}
$$

Finally, the integral is calculated as:

$$
\begin{aligned}
\iint_S (x-y+z^2)\, dS &= \iint_S (t-s+9s^2)\, 3\sqrt{6}\, ds\, dt \\
&= 3\sqrt{6} \int_0^1 \int_0^{1-t} t-s+9s^2\, ds\, dt \\
&= 3\sqrt{6} \int_0^1 \left[ts - \frac{s^2}{2} + 3s^3 \right]_0^{1-t} dt \\
&= 3\sqrt{6} \int_0^1 t - t^2 - \frac{(1-t)^2}{2} + 3(1-t)^3\, dt \\
&= 3\sqrt{6} \int_0^1 t - t^2 - \frac{(1-t)^2}{2} + 3(1-t)^3\, dt \\
&= 3\sqrt{6} \int_0^1 t - t^2 + \left(-\frac{1}{2} + t - \frac{t^2}{2} \right) + \left(3 - 9t + 9t^2 - 3t^3 \right)\, dt \\
&= 3\sqrt{6} \frac{3}{4} \\
&= \frac{9\sqrt{6}}{4}
\end{aligned}
$$

Exercise 4.7 Use Stokes' Theorem to evaluate $\int_C F \cdot d\mathbf{r}$ where

$$F(x,y,z) = xy\,\mathbf{i} + xz\,\mathbf{j} + x^2\,\mathbf{k}$$

and $C = \partial R$ with R the Rhombus with corners at $(0,0,0), (5,0,5), (5,5,10)$ and $(0,5,5)$, and has positive orientation when viewed from above. ■

First notice that the plane that intersects all 4 points is defined by

$$-x - y + z = 0$$

Again, this is now a simple calculation:

$$
\begin{aligned}
\int_C F \cdot d\mathbf{r} &= \iint_S \text{curl}(F) \cdot \hat{n}\, dS \\
&= \iint_S (\nabla \times F) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dx\,dy \\
\mathbf{r}(x,y) &= \langle x, y, x+y \rangle \\
\mathbf{r}_x &= \langle 1, 0, 1 \rangle \\
\mathbf{r}_y &= \langle 0, 1, 1 \rangle \\
\mathbf{r}_x \times \mathbf{r}_y &= \langle -1, -1, 1 \rangle \\
\text{curl}(F) = \nabla \times F &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & x^2 \end{bmatrix} \\
&= \langle -x, -2x, z-x \rangle \\
\iint_S (\nabla \times F) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dx\,dy &= \int_0^5 \int_0^5 \langle -x, -2x, z-x \rangle \cdot \langle -1, -1, 1 \rangle\, dx\,dy \\
&= \int_0^5 \int_0^5 x + 2x + z - x\, dx\,dy \\
&= \int_0^5 \int_0^5 2x + z\, dx\,dy \\
&= \int_0^5 \int_0^5 3x + y\, dx\,dy \\
&= \int_0^5 \frac{75}{2} + 5y\, dy \\
&= \left[\frac{75}{2}y + \frac{5}{2}y^2 \right]_0^5 \\
&= 250
\end{aligned}
$$

R The unit normal can be verified by seeing that $\mathbf{r}_x \times \mathbf{r}_y$ is orthogonal to every point on the Rhombus, that is, $(-1,-1,1) \cdot p = 0$ for any $p \in R$. Also note that the bounds for the integration is found by viewing R from above, which makes a square in the first quadrant of the $xy-$plane, with sides of length 5.

4.2 Comments

This semester was essentially a recap of high-school multivariate calculus. For exercise 3.1 see that a triple integral gives you the **volume** of the solid when you integrate over 1, with dV being the appropriate Jacobian. If you take some real-valued function $f(x,y,z)$ we are assigning to each point $(x,y,z) \in \mathbb{R}$ some value. The triple integral adds all these up, giving some total "density". For example, if the function was a constant c, then the triple integral would simply be cA where A is the area.

R For exercises 3.1-3.5 please note that I drew pictures to figure out the total domain of each chart, but I did **not** include them in this document. When doing these and similar problems please don't forget to draw pictures. They give intuition and make problem solving effortless.

A solid understanding of the material recapped will help in differential equations and differential geometry.

4.3 Notes and concepts to remember

The following are some notes to remember. Keep in mind these follow naturally from the previous ones, so that the reader must understand each one in succession.

Definition 4.3.1 A **vector field** in $\mathbb{R}^3$ is a function that assigns to each point $p \in \mathbb{R}^3$ a vector. We denote this:

$$F(x,y,z) = F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k} = \langle F_1, F_2, F_3 \rangle$$

Recall that the *gradient* ∇f is given by

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Definition 4.3.2 A vector field F is a **conservative vector field** if and only if it is of the form $F = \nabla f$ where f is a real valued function. Here f is called the **potential** of F.

R For some functions it might be very difficult to try and deduce the potential function. For this we use the *curl test*. Look ahead for the definition.

Definition 4.3.3 Given a continuous vector field $F(x,y,z)$ and a path C parametrized by $\mathbf{r}(t)$ with $a \leq t \leq b$ then the **line integral** of F over C is defined as

$$\int_a^b F \cdot d\mathbf{r}$$

where $d\mathbf{r} = \mathbf{r}'(t)\,dt$.

As the usual with calculus there are many different notations for the same thing. The most common alternative notation for the above line integral is the *differential form* notation. Here if $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ then with some abuse of notation we can "derive"

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

so that for some vector field $F = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ we can "derive"

$$F \cdot d\mathbf{r} = \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle = \langle P\,dx, Q\,dy, R\,dz \rangle$$

thus we can use this notation to represent:

$$\int_a^b F \cdot d\mathbf{r} = \int_a^b P\,dx + Q\,dy + R\,dz$$

Theorem 4.3.1 Given a conservative vector field $F = \nabla f$, and a path C with starting point (x_0, y_0, z_0) and endpoint (x_1, y_1, z_1) then

$$\int_C F \cdot d\mathbf{r} = f(x_0, y_0, z_0) - f(x_1, y_1, z_1)$$

This is amazing! What it says, is that the line integral can be calculated just from knowing the endpoints. This implies the line integral of *any curve with the same endpoints* over a conservative vector field will be the same. What if the curve is a loop that ends on the same point that it started at?

Theorem 4.3.2 On a conservative vector field F with a closed curve C we have:

$$\oint F \cdot d\mathbf{r} = 0$$

Definition 4.3.4 — Curl. Given a vector field

$$F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

the curl of the vector field F, denoted $\nabla \times F$, is defined as:

$$\begin{aligned}\nabla \times F &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\end{aligned}$$

Theorem 4.3.3 — Curl Test. Given a C^1 vector field F defined on a simply connected region Ω, then we have the following:

- $\Longrightarrow$) If $F = \nabla f$ for some (potential) scalar function f on Ω, then $\nabla \times F \equiv \mathbf{0}$ on Ω.
- $\Longleftarrow$) If $\nabla \times F \equiv \mathbf{0}$ and Ω is simply connected, then there exists a function f such that $F = \nabla f$.

Note above that we are essentially stating: "If F is a conservative vector field if and only if the curl of F is 0". But we have split it in the forward direction and the reverse. The forward is simply a consequence of the Mixed Partial Theorem, but the converse is very technical to prove. **Please note:** The curl test only shows that the vector field is conservative, and that there *exists* a potential function, *it doesn't say what the potential function is*. The use is that we don't have to integrate and deduce the potential, but we can simply use differentiation.

Definition 4.3.5 — Surface Integral. Given a surface Ω parametrized by $\mathbf{r}(u,v)$ in $\mathbb{R}^3$ and a *scalar valued function* $f(x,y,z)$ defined on Ω we have

$$\iint_\Omega f\, d\Omega = \iint_\Omega f(\mathbf{r}(u,v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du\, dv$$

The above function assigns to each point on the surface Ω a value and multiplies it by the area element, then sums all of those up. It gives the total surface mass. To find the surface area, you can just let $f(x,y,z) = 1$ to get:

$$\text{Area}(\Omega) = \iint_\Omega 1\, d\Omega = \iint_\Omega \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du\, dv$$

Theorem 4.3.4 — Stokes' Theorem. Let S be an oriented smooth surface in $\mathbb{R}^3$ that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Let F be a vector field. Then

$$\oint_C F \cdot dr = \iint_S \text{curl}(F) \cdot \hat{\mathbf{n}}\, dS$$

Green's theorem is a lower dimensional analogue to Stokes' theorem. Here if you have some vector field $F = P\mathbf{i} + Q\mathbf{j}$ then

$$\text{curl}(F) = \nabla \times F = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{bmatrix} = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

and *since the normal vector to the plane* is just $\hat{\mathbf{n}} = \langle 0, 0, 1 \rangle$ we have the 2d case of Stokes theorem:

Theorem 4.3.5 — Greens' Theorem. Let C be a simple closed curve in $\mathbb{R}^2$ which is counter clockwise oriented (right hand rule). Then if C encloses a reigon R, and $F = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a vector field then

$$\oint_C F \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Finally to recap remember we have

- **Gradient** of a scalar-valued function gives a vector field defined by:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

- **Divergence** of a vector field $F = (F_1, F_2, F_3)$ gives a scalar function defined by:

$$\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- **Curl** of a vector field $F = (F_1, F_2, F_3)$ gives another vector field defined by:

$$\nabla \times F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$