COMPACT SELF-EXPANDERS TO GENERALIZED INVERSE CURVATURE FLOWS

ISAAC ABBOTT

Abstract. In this paper, we study self-expanding solutions for a broad class of parabolic inverse curvature flows. These flows are governed by homogeneous symmetric functions of the principal curvatures in Euclidean spaces. We show that the only compact self-expanders for any of these flows are round spheres, generalizing results from [4] and [12] using ideas from [3].

1. Introduction

The curve shortening flow in the plane is the most natural way to smooth out a closed curve. Each point of the curve moves in the normal direction with speed equal to its curvature. Intuitively, regions of high curvature contract faster, and the evolution tends to smooth out irregularities. This process reduces length most efficiently. The mean curvature flow is the natural generalization of curve shortening to higher-dimensional surfaces. A hypersurface Σ^n in $\mathbb{R}^{n+1}$ evolves by moving each point in the normal direction with speed equal to its mean curvature. Just as curvature flow in the plane decreases length most efficiently, mean curvature flow decreases surface area most efficiently.

The first notable study of mean curvature flow for convex hypersurfaces was conducted by Huisken in 1984 [8]. Subsequently, Huisken and Ilmanen investigated the inverse mean curvature flow and employed it to prove the Riemann–Penrose inequality [9]. Since then, the inverse mean curvature flow has been a topic of interest, both for its applications and for it's own sake. Drugan, Lee, and Wheeler in 2016 showed that the only compact self-expanders to the inverse mean curvature flow are round spheres [4]. This was generalized later to other curvature flows by [11] and [2].

Please note that this is a private draft, so a lot of sentences are missing, and sections will be repetitive due to the order in which I have written this draft.

2. Question and Structure

Before proceeding to the preliminaries in Section 3, it is useful to state the main question clearly. The mean curvature flow evolves each point $\vec{x}$ on a hypersurface Σ in the direction of the inward-pointing unit normal ν with speed equal to the mean curvature H at that point. This evolution is governed by the equation

$$\frac{\partial}{\partial t}\vec{x} = -H\nu.$$

Huisken proved in 1984 that every strictly convex hypersurface contracts to a round point under this flow [8]. Here a hypersurface is strictly convex if all principal curvatures satisfy $\lambda_i > 0$ at every point $p \in \Sigma^n$ and for all $i \in \{1, \ldots, n\}$. Huisken later extended the result to a weaker convexity condition: any 2-convex hypersurface with $\lambda_1 + \lambda_2 \geq 0$ also converges to a sphere.

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We now turn to the inverse mean curvature flow (IMCF). In contrast to the mean curvature flow, each point $\vec{x}$ on the hypersurface moves in the outward normal direction with speed inversely proportional to the mean curvature H. The evolution equation is

$$\frac{\partial}{\partial t}\vec{x} = \frac{1}{H}\nu,$$

where ν denotes the outward-pointing unit normal.

This inverse flow, along with its generalisations, was first studied independently by Gerhardt [7] and Urbas [14]. They showed that any initial compact star-shaped hypersurface evolves under the flow into a round sphere. Notice the title of Urbas' paper:

> "On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures"

This hints at the fact the MCF is one in a class of many other extrinsic flows. The mean curvature is given by $\lambda_1 + \lambda_2$, while the Gauss curvature is $\lambda_1\lambda_2$. Both are symmetric functions of the principal curvatures. It is therefore natural to consider a general symmetric function $\rho(\lambda_1, \ldots, \lambda_n)$ and define the inverse ρ-flow by

$$\frac{\partial}{\partial t}\vec{x} = -\frac{1}{\rho(\lambda_1, \ldots, \lambda_n)}\nu.$$

This leads to the following natural question.

Question 1. Does any initial star-shaped convex hypersurface that satisfies the inverse ρ-flow as defined above converge to a sphere?

The answer to this question is no. A counterexample appears in the appendix. To obtain convergence to a sphere, further restrictions must be imposed on the function ρ. The precise class of admissible functions ρ is given in the following definition.

Definition 1 (Inverse ρ-Flow)**.** Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a smooth, closed hypersurface and let $F : \Sigma^n \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth family of embeddings. This family is called an **inverse ρ-flow** if it satisfies the evolution equation

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \frac{1}{\rho(\lambda_1, \ldots, \lambda_n)}\nu,$$

where ν is the evolving unit outward normal and ρ is a function of the principal curvatures λ_i defined on an open symmetric cone $\Gamma \subset \mathbb{R}^n$ that satisfies the following four conditions:

- *Symmetry:* ρ is a symmetric function of $(\lambda_1, \ldots, \lambda_n)$. This ensures that the flow is independent of the ordering of the principal directions.
- *Homogeneity of degree one:* For any $c > 0$, $\rho(c\lambda_1, \ldots, c\lambda_n) = c\rho(\lambda_1, \ldots, \lambda_n)$.
- *Positivity:* $\rho > 0$ on Γ, and Γ contains the positive cone $\{\lambda_i > 0 : \forall i\}$.
- *Monotonicity:* ρ is of class C^1 and satisfies
  $$\frac{\partial \rho}{\partial \lambda_i} > 0 \quad \text{on } \Gamma$$
  for all $i = 1, \ldots, n$.

Many examples of such ρ-flows exist; a short list may be found in [2]. With this class of flows in hand, a second natural question arises.

Question 2. Does any arbitrary initial surface asymptotically approach a spherical shape as it expands under *any* inverse ρ-flow?

The answer is yes! Chow, Chow, and Fong established this result in [2] as theorem 4.1. To state their theorem they first introduced the notion of a self-expander.

Definition 2 (Self-Expander)**.** A smooth, closed hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is called a **self-expander** to the inverse ρ-flow if there exists a constant $\mu > 0$ such that the following stationary equation holds:

$$-\frac{1}{\rho(\lambda_1, \dots, \lambda_n)} = \mu\langle F, \nu\rangle,$$

where F is the position vector, ν is the unit outward normal, and $\langle F, \nu\rangle$ denotes the support function of the hypersurface.

They then proved the following theorem.

Theorem 3 (Chow, Chow, Fong)**.** *Consider an inverse ρ-flow governed by a C^1 symmetric function of the principal curvatures $\rho(\lambda_1, \dots, \lambda_n)$, defined on an open cone Γ such that $(1, \dots, 1) \in \Gamma$. Suppose that ρ is homogeneous of degree one and satisfies the ellipticity condition $\frac{\partial \rho}{\partial \lambda_i} > 0$ for all $i = 1, \dots, n$ on Γ. Then the only compact hypersurfaces that arise as self-expanders to this flow are the round spheres.*

The first half of this article provides a complete walk-through of the proof of Theorem 3. The second half examines a further question.

Question 3. Is it possible to weaken some requirement in the definition of the inverse ρ-flow while still preserving the conclusion of Theorem 3?

We conjecture that the answer is yes. One promising direction is to relax the homogeneity of degree one condition. Another is to weaken the convexity requirement on the initial surface. In particular, if every compact self-expander can be shown to become convex after a short time, then round spheres would remain the only self-expanders. This would extend the theorem to all star-shaped initial surfaces and eliminate the need for an a-priori convexity assumption on the cone Γ.

3. Preliminaries

Let's first recap some notation. If $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$, then we define

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle$$
$$A_{ij} = \left\langle \frac{\partial F}{\partial x_i \partial x_j}, \nu \right\rangle$$

which are the first and second fundamental forms of the hypersurface Σ parametrized by F. We let $[g]$ and A be the matrices defined by g_{ij} and A_{ij}, respectively. We let g^{ij} denote the elements of $[g]^{-1}$. Then further define

$$A_i^j := g^{jk} A_{ik}$$

which allows us to define $H = A_i^i = g^{ij} A_{ij}$ as the mean curvature. It is important to see that the principal curvatures $\lambda_1 \le \cdots \le \lambda_n$ are the eigenvalues of the matrix $[A_i^j]$. It will be useful to let

$$|A|^2 = g^{ij} g^{lk} A_{il} A_{jk} = \lambda_1^2 + \cdots + \lambda_n^2$$

This will be important for the cylindrical estimates we will soon develop. A hypersurface is said to be **convex** if for all i we have $\lambda_i \ge 0$. It is called **strictly-convex** if $\lambda_i > 0$. It is called **mean-convex** if $H = \lambda_1 + \cdots + \lambda_n \ge 0$. Before moving on, we will see some examples of computations.

Example 4 (Graph of a function)**.** Suppose the hypersurface $\Sigma_f \subset \mathbb{R}^{n+1}$ is locally a graph of a function $f : \mathbb{R}^n \to \mathbb{R}$. That is, $F(x) = (x, f(x))$, with $x = (x_1, \ldots, x_n)$.

$$\begin{aligned}
\frac{\partial F}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(x_1, \ldots, x_i, \ldots, x_n, f(x)\right) \\
&= \left(0, \ldots, 1, \ldots, \frac{\partial f}{\partial x_i}\right) \\
&= e_i + \frac{\partial f}{\partial x_i} e_{n+1} \\
g_{ij} &= \left\langle e_i + \frac{\partial f}{\partial x_i} e_{n+1}, e_j + \frac{\partial f}{\partial x_j} e_{n+1} \right\rangle \\
&= \delta_{ij} + \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j}
\end{aligned}$$

so in matrix notations

$$[g] = I_{n \times n} + (\nabla f)(\nabla f)^T$$

which is the first fundamental form. Now we calculate the second fundamental form. To do this, we will first need to calculate the Gauss map. Using our term for $\frac{\partial F}{\partial x_i}$ we are able to see

$$\begin{aligned}
\frac{\partial^2 F}{\partial x_i \partial x_j} &= \frac{\partial F}{\partial x_i}\left(e_j + \frac{\partial f}{\partial x_j} e_{n+1}\right) \\
&= \frac{\partial^2 f}{\partial x_i \partial x_j} e_{n+1}
\end{aligned}$$

and one can take a normal component $N = (-\nabla f, 1)$ which is indeed orthogonal to $\frac{\partial F}{\partial x_i}$ by

$$\left\langle N, \frac{\partial F}{\partial x_i} \right\rangle = (-\nabla f, 1) \cdot \left(e_i + \frac{\partial f}{\partial x_i} e_{n+1}\right) = -\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_i} = 0$$

finding the norm of this $\|N\| = \sqrt{1 + |\nabla f|^2}$ allows us to normalize N, then find the Gauss map like so:

$$\nu = \frac{N}{|N|} = \frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}$$

so this gives us the second fundamental form

$$\begin{aligned}
A_{ij} &= \left\langle \frac{\partial F}{\partial x_i \partial x_j}, \nu \right\rangle \\
&= \frac{\partial^2 f}{\partial x_i \partial x_j} e_{n+1} \cdot \frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|}} \\
&= \frac{\frac{\partial^2 f}{\partial x_i \partial x_j}}{\sqrt{1 + |\nabla f|}}
\end{aligned}$$

or in matrix notations:

$$A = \frac{\nabla^2 f}{\sqrt{1 + |\nabla f|^2}}$$

where $\nabla^2 f$ is the Hessian of f, whose (i, j)-th entry is given by $\frac{\partial^2 f}{\partial x_i \partial x_j}$. Now to find the mean curvature $H = g^{ij} A_{ij}$ by finding inverse of $[g]$ (add this working soon), then we are able to see

$$H = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right)$$

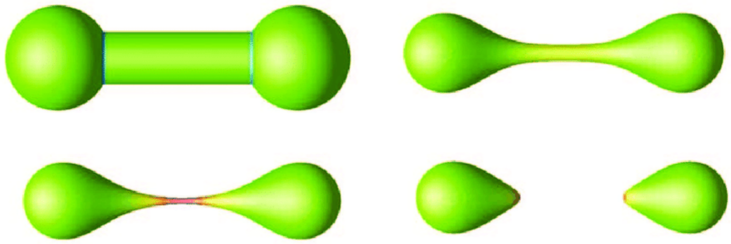


FIGURE 1. Dumbbell developing singularity under MCF

Example 5 (Surface of revolution in $\mathbb{R}^3$)**.** Add this soon.

4. Mean curvature flow

We now introduce the mean curvature flow based on what we have established. Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a closed hypersurface, and $F : \Sigma^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth-family of embedded surfaces. From now on we will denote this by Σ_0 being the *initial* surface, and $\{\Sigma_t \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ to be the smooth family of embedded surfaces. Let $\Sigma_t := F(\Sigma, t)$. Then this family is called a **mean curvature flow**, with initial surface Σ_0 if the following hold:

$$\begin{cases} \frac{\partial F}{\partial t} = -H\nu \\ F(p,0) = p \quad p \in \Sigma_0 \end{cases}$$

where ν is the outward pointing Gauss map. This can be visualized as taking a point, and moving inwards proportional to its mean curvature.

Remark 6. There are some sign conventions. Some of the articles in the literature take ν to be the inward pointing Gauss map, so that they can say $\partial_t F = H\nu$. Some even let $\vec{H} := H\nu$ so they can say $\partial_t F(p) = \vec{H}(p)$.

Before we compute an example let's take a look at some cases. Given $H = 0$ at every point on the surface, no points will move anywhere, and the initial surface will stay stationary. This is called a minimal surface, and is a topic of much interest for its applications (a bubble is a minimal surface since it tries to minimize its surface area). Suppose $H > 0$ at some point $p \in \Sigma_0$, then over time this p will move inwards, till $H = 0$. Likewise, if $H < 0$ at a point, then the point will move outwards. This is at a speed according to the magnitude of H. So, the surface evolves in such a way that regions with higher curvature shrink faster, while flatter regions move slower. This can be visualized as the initial surface having its sharpest bumps smoothed out first, with its smaller bumps smoothing out over time. As one can imagine, it would seem that any shape eventually turns into a sphere. Unfortunately, this is only true given some restrictions. Husiken proved in [8] that any *strictly convex* hypersurface eventually becomes a sphere. If the initial surface is not convex, it may develop *singularities*. An example is the dumbbell.

As you can see in figure 1, the mean curvature works faster on the cylindrical part, than the bulbs. This causes a singularity in between the two, in which the dumbells break off (we will present this proof soon). As you can imagine, this doesn't behave nicely as an ODE, so there have been many techniques developed to fix this problem. We list some common ones:

- Geometric Surgery. Surgery techniques were initially developed by Hamilton in the settings of 3-manifolds, with aims to solve the Poincare conjecture. This was utilized by Husiken-Sinestrari in [10] in the context of mean curvature flow of 2-convex surfaces ($\lambda_1 + \lambda_2 \geq 0$). The idea is to cut the surface at the thin neck, when the singularity forms, and cap the ends with standard spherical caps, and then continue the flow. This allowed Husiken-Sinestrari to prove that any 2-convex surface evolving under MCF only requires finite surgeries before each remaining component can be classified.
- Level-Set Flow. Instead of viewing the family $\{\Sigma_t \subset \mathbb{R}^{n+1}\}$ as a smooth family of embeddings, we can describe the evolving hypersurface as a level set $u(p, x) = 0$ for $p \in \Sigma_t$. This was pioneered by Evans and Spruck in 1992 [6]. This method meant intervention methods such as geometric surgery were not needed, since the level sets can flow through singularities smoothly. I think this has it's limitations (still learning about it after taking DE's).
- Brakke Flow. This takes a measure theoretic approach, so this author cannot comment on this, only that it must be quite useful because it can be seen everywhere in the literature.

While some see singularities as a problem, some see them as a topic of interest. Much work has been done to classify these singularities. We have already shown that given the convexity assumption, Husiken proved the singularities are modeled by spheres [8]. White then showed, that given mean convexity, any singularities are modeled by spheres or cylinders [15, 17, 16]. Here "modeled" refers to the process of rescaling the surface (this is actually quite difficult as we will see) immediately before it collapses to a point, in order to see its shape. With no curvature assumptions, the classification of singularities that arise in MCF remains widely open. The study of these singularities, the time they occur, and the asymptotic behavior of solutions near singularities is broadly referred to as regularity theory (see [5]). Husiken showed that these singularities are modeled by a special class of surfaces called *self-shrinkers*. Self-shrinkers and self-expanders are families of surfaces $\{\Sigma_t\}$ that evolve under mean curvature flow in such a way that, so that at each time t, the surface is a scaled version of the initial surface Σ_0. Visually, the surface evolves, but keeps its initial shape. As promised, before we talk about this, we will see some computations.

Example 7 (Sphere)**.** Let Σ_0 be a sphere. Then we will show the family of cocentric spheres $\Sigma_t := \partial B^{n+1}_{r(t)}(0)$ is a mean curvature flow, for a radius function $r(t)$. Take any point p on the sphere. Then by definition $\nu(p, t) = p$. The mean curvature at a point on the sphere is the sum of the principal curvatures at that point. Since all these principal curvatures are the same on a sphere, it suffices to find one. By definition of curvature, this is $\lambda_i = 1/r$. For an $n-$dimensional sphere with radius $r(t)$ at time t, the mean curvature is

$$H = \overbrace{\frac{1}{r(t)} + \cdots + \frac{1}{r(t)}}^{n-\text{times}} = \frac{n}{r(t)}$$

Thus by letting $F(p,t) = r(t)p$ we calculate

$$\frac{\partial F}{\partial t} = -H\nu$$
$$\frac{\partial}{\partial t} r(t)p = -\frac{n}{r(t)}p$$
$$\frac{\partial}{\partial t} r(t) = -\frac{n}{r(t)}$$

and solving this ODE yields:

$$\frac{dr}{dt} = -\frac{n}{r}$$
$$r\,dr = -n\,dt$$
$$\frac{r^2}{2} = C_1 - nt$$
$$r = \sqrt{C - 2nt}$$

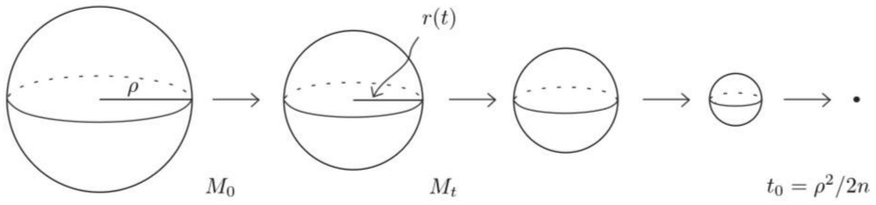
notice because the mean curvature is the same everywhere, it is reduced to an ODE. Given an initial radius $r(0) = R$, we see

$$r(0) \implies R = \sqrt{C} \implies C = R^2$$

giving us a solution to the MCF:

$$r(t) = \sqrt{R^2 - 2nt}$$

this solution can be visualized in $\mathbb{R}^3$ as so:



the sphere collapses to a singularity at time $t = R^2/2n$. This is an example of a self-shrinker.

Example 8 (Shrinking Cylinder)**.** Show example of cylinder. Show it is self-shrinker. But collapses to a line.

Example 9 (Shrinking Torus)**.** Show the example of a torus. Show it collapses to a circle. Show that this is *not* a self-shrinker.

Theorem 10 (Avoidance Principle)**.** *The avoidance principle says that, roughly speaking, under the mean curvature flow two disjoint embedded hypersurfaces will never intersect with each other. If Σ_t^1 and Σ_t^2 are two mean curvature flows with $\Sigma_0^1 \cap \Sigma_0^2 = \emptyset$ then $\Sigma_t^1 \cap \Sigma_t^2 = \emptyset$ for all t. (show proof as in [13]).*

We have enough to prove an interesting theorem, that a singularity does occur in the dumbbells as showed in figure 1. When we gave the example of the torus, we showed that it was *not* a self-shrinker. This is because its appearance changes as it evolves under mean curvature flow. Angenent in [1] constructed a torus that is a self-shrinker. That is, it maintains the same shape under evolution, and merely shrinks. This can be visualized in figure 2. Put an Angenent torus inside the neck region as a barrier. This torus has to be a "suitable" size enough to fit into either bulb. This is so we know the torus must shrink to a point before the

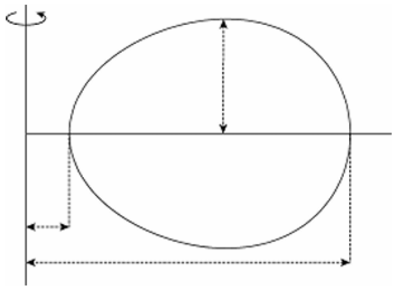


FIGURE 2. The graph rotated around the vertical axis to create Angenent's torus

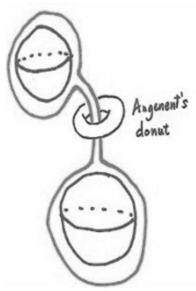


FIGURE 3. Visualization of Proof

bulbs do, by avoidance principle (picture this torus inside a sphere under MCF). Because the torus shrinks self-similarly and the two bulb regions of the dumbbell are on different sides of the torus's hole, avoidance forces the dumbbell to develop a singularity (pinching) before a possible smooth process that would let the two bulbs pass through the torus region. In other words, the Angenent torus blocks the neck from disappearing smoothly, so the flow *must singularize* (as shown in [1]).

5. Variational Formulae

We use this section to develop several well known variational formulae to use in our proofs. Let's first explore a familiar proposition:

Proposition 11. Let $\Sigma_t \subseteq \mathbb{R}^3$ be a family of hypersurfaces for $t \in (-\epsilon, \epsilon)$, with each parametrization:

$$F_t(u_1, u_2) := F_0(u_1, u_2) + t\varphi(u_1, u_2)\upsilon(u_1, u_2)$$

where $\upsilon(u_1, u_2)$ is the Gauss map at $F_0(u_1, u_2)$, and $\varphi(u_1, u_2)$ is a smooth function on Σ_0. Then

$$\left.\frac{d}{dt}\right|_{t=0} g_{ij}(t) = -2\varphi h_{ij}(t)$$

Proof. We start by calculating the tangent vectors of the varied surface F_t:

$$\frac{\partial F_t}{\partial u_i} = \frac{\partial F_0}{\partial u_i} + t\left(\frac{\partial \varphi}{\partial u_i}\upsilon + \varphi\frac{\partial \upsilon}{\partial u_i}\right)$$

Now, we express the metric tensor $g_{ij}(t) = \left\langle \frac{\partial F_t}{\partial u_i}, \frac{\partial F_t}{\partial u_j} \right\rangle$. Expanding the inner product:

$$g_{ij}(t) = \left\langle \frac{\partial F_0}{\partial u_i} + t\left(\frac{\partial \varphi}{\partial u_i}\upsilon + \varphi\frac{\partial \upsilon}{\partial u_i}\right), \frac{\partial F_0}{\partial u_j} + t\left(\frac{\partial \varphi}{\partial u_j}\upsilon + \varphi\frac{\partial \upsilon}{\partial u_j}\right)\right\rangle$$
$$= \left\langle \frac{\partial F_0}{\partial u_i}, \frac{\partial F_0}{\partial u_j} \right\rangle + t\left[\left\langle \frac{\partial F_0}{\partial u_i}, \frac{\partial \varphi}{\partial u_j}\upsilon + \varphi\frac{\partial \upsilon}{\partial u_j}\right\rangle + \left\langle \frac{\partial \varphi}{\partial u_i}\upsilon + \varphi\frac{\partial \upsilon}{\partial u_i}, \frac{\partial F_0}{\partial u_j}\right\rangle\right] + O(t^2)$$

Differentiating with respect to t and evaluating at $t = 0$, the $O(t^2)$ terms vanish:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} g_{ij}(t) = \left\langle \frac{\partial F_0}{\partial u_i}, \frac{\partial \varphi}{\partial u_j}\upsilon + \varphi\frac{\partial \upsilon}{\partial u_j}\right\rangle + \left\langle \frac{\partial \varphi}{\partial u_i}\upsilon + \varphi\frac{\partial \upsilon}{\partial u_i}, \frac{\partial F_0}{\partial u_j}\right\rangle$$

Because the Gauss map υ is normal to the tangent plane, $\left\langle \frac{\partial F_0}{\partial u_i}, \upsilon \right\rangle = 0$. This simplifies the expression to:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} g_{ij}(t) = \varphi\left\langle \frac{\partial F_0}{\partial u_i}, \frac{\partial \upsilon}{\partial u_j}\right\rangle + \varphi\left\langle \frac{\partial \upsilon}{\partial u_i}, \frac{\partial F_0}{\partial u_j}\right\rangle$$

Using the relation $h_{ji} = h_{ij} = -\left\langle \frac{\partial F_0}{\partial u_i}, \frac{\partial \upsilon}{\partial u_j} \right\rangle$

$$\left.\frac{\partial}{\partial t}\right|_{t=0} g_{ij}(t) = \varphi(-h_{ij}) + \varphi(-h_{ji}) = -2\varphi h_{ij}$$

□

What we have shown is that for an initial hypersurface perturbed by some smooth function we can find the derivative of the metric tensor as a relation of the second fundamental form. We can easily generalise this proposition to a smooth family of hypersurfaces $F_s : \Sigma^n \to \mathbb{R}^{n+1}$ that satisfy...

Next steps:

- Derive the geometric evolution equations to prove everything said, but formally.
- Give formal notion of a self-similar solution = self-shrinker + self-expander.
- Show self-expander is just important as a self-shrinker.
- Develop inverse mean curvature flow.
- Show that the only compact self expanders to IMCF are round spheres.
- Develop other IMCF such as Q_k, symmetric, guass, etc flows.
- Show that these can be all particular forms of a generalized IMCF.
- Show that the only self-expander to these generalized curvature flows are round spheres.

6. Notation and Preliminaries

Let's first recall some things from Euclidean geometry. Given some curve that is parametrized by $\gamma(t) : I \to \mathbb{R}^2$, we define the tangent plane as $T(s) = \gamma'(t)$. Instead of defining the normal frame $N(t) = \frac{1}{\kappa(t)}T(t)$, we use the frame $JT(t)$ where J is is the counter-clockwise rotation by $\frac{\pi}{2}$ defined by

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

we can now define the signed curvature function.

Definition 12. Give an arc-length parameterized plane curve $\gamma(t) : I \to \mathbb{R}^2$, we define the **signed curvature** $k(t) : I \to \mathbb{R}$

$$k(t) := T'(s) \cdot JT(s)$$

Let's compute the signed curvature of the unit circle $\gamma(t) = (\cos(t), \sin(t))$ by seeing that $T(s) = (-\sin(t), \cos(t))$ so therefore $JT = (-\cos(t), -\sin(t))$. Then in total we have

$$\begin{aligned} T(s) &= (-\sin(t), \cos(t)) \\ k(t) &= T'(t) \cdot JT(t) \\ &= 1 \end{aligned}$$

Notice a few things. First, the signed curvature is the same regardless of how the curve is oriented. Secondly, the speed of the curve doesnt matter: we will get the same answer for $\gamma(t) : (0, 2\pi) \to \mathbb{R}^2$ as we do for $\gamma(t) : (0, 1) \to \mathbb{R}^2$. We will now use this to examine the mean curvature. First take some hypersurface defined by $F : \mathbb{R} \to \Sigma$. Then take some $p \in \Sigma$, then the normal curvature at p in the direction

7. The main part

In this part (today is February 16th, 2025) we will first study the first and second fundamental forms of various Euclidean hypersurfaces $\Sigma^{n\geq 2} \subseteq \mathbb{R}^{n+1}$. We will use both matrix and tensor notations. For such a regular hypersurface with $F(u_i) : \mathcal{U} \subseteq \mathbb{R}^n \to \Sigma$ as one of its local parametrizations, we define the Gauss map at $p \in \Sigma$ to be the orthogonal complement of $T_p\Sigma$. This is a 1-dimensional subspace of $\mathbb{R}^{n+1}$, and we will denote this by $N_p\Sigma$. Take a smooth function $f(u_1, \ldots, u_n)$ whose graph is the subset

$$\Sigma^n := \{(u_1, \ldots, u_n, f(u_1, \ldots, u_n)) : (u_1, \ldots, u_n) \in \mathcal{U}\} \subseteq \mathbb{R}^{n+1}$$

Lemma 13. *The graph of a smooth function is a hypersurface.*

Proof. The surface Σ^n can be globally parametrized by a single parametrization given by

$$F(u_1, \ldots, u_n) = (u_1, \ldots, u_n, f(u_1, \ldots, u_n)) \tag{7.1}$$

It can be shown

$$\frac{\partial F}{\partial u_i} = \hat{e}_i + \frac{\partial f}{\partial u_i}\hat{e}_{n+1} \tag{7.2}$$

where $\{\hat{e_1}, \ldots, \hat{e_n}\}$ are the basis vectors of $\mathbb{R}^n$. The lemma follows after seeing $\{\frac{\partial F}{\partial u_i}\}$ are linearly independent. □

8. Notation and Preliminaries

We can now use this to prove the main theorem.

Theorem 14. *Let $f(u_1, \ldots, u_n) : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a smooth function defined on an open set $\mathcal{U}$.*

(1) *The first fundamental form of the hypersurface graphed by f can be expressed as*
$$[g] = I_{n\times n} + (\nabla f)(\nabla f)^\top.$$
(2) *The second fundamental form of the hypersurface graphed by f can be expressed as*
$$[h] = \frac{\nabla^2 f}{\sqrt{1 + |\nabla f|^2}}.$$

Recall 7 points out something important (citation test). We then prove the following theorem.

Theorem 15. *Let $\Sigma_t \subseteq \mathbb{R}^3$ be a family of hypersurfaces for $t \in (-\epsilon, \epsilon)$, with each parametrization:*

$$F_t(u_1, u_2) := F_0(u_1, u_2) + t\varphi(u_1, u_2)\upsilon(u_1, u_2)$$

where $\upsilon(u_1, u_2)$ is the Gauss map at $F_0(u_1, u_2)$, and $\varphi(u_1, u_2)$ is a smooth function on Σ_0. Then

(1)
$$\left.\frac{d}{dt}\right|_{t=0} g_{ij}(t) = -2\varphi h_{ij}(t)$$

(2)
$$\left.\frac{d}{dt}\right|_{t=0} \log\det[g_{ij}(t)] = -2H$$

9. Proof of Theorem 15

We will start by calculating the first and second fundamental forms g_{ij} and h_{ij}:

$$\begin{aligned}
\frac{\partial F_t}{\partial u_i} &= \frac{\partial F_0}{\partial u_i} + t\frac{\partial \varphi}{\partial u_i}\upsilon + t\frac{\partial \upsilon}{\partial u_i}\varphi \\
g_{ij}(t) &= \left\langle \frac{\partial F_0}{\partial u_i} + t\frac{\partial \varphi}{\partial u_i}\upsilon + t\frac{\partial \upsilon}{\partial u_i}\varphi, \frac{\partial F_0}{\partial u_j} + t\frac{\partial \varphi}{\partial u_j}\upsilon + t\frac{\partial \upsilon}{\partial u_j}\varphi \right\rangle \\
&= g_{ij}(0) + t\left\langle \frac{\partial F_0}{\partial u_i}, \frac{\partial \varphi}{\partial u_j}\upsilon \right\rangle + t\left\langle\right\rangle
\end{aligned}$$

After differentiation, and using the relation $h_{ij} = h_{ji} = -\left\langle \frac{\partial F_0}{\partial u_i}, \frac{\partial \upsilon}{\partial u_j} \right\rangle$

$$\left.\frac{\partial}{\partial t}\right|_{t=0} g_{ij}(t) = \left\langle \frac{\partial F_0}{\partial u_i}, \frac{\partial \upsilon}{\partial u_j} \right\rangle + \left\langle \frac{\partial F_0}{\partial u_j}, \frac{\partial \upsilon}{\partial u_i} \right\rangle = -2h_{ij}$$

Lemma 16. *For any i, j, k, q we have*

$$R^q_{ijk} = g^{q\lambda}(h_{jk}h_{\lambda i} - h_{ik}h_{\lambda j})$$

Proof. By Gauss's equation. □

Theorem 17 (Gauss)**.** *For any regular surface in $\mathbb{R}^3$ then K depends only on g.*

Proof. By Lemma 16 we have for any i, j, k, q

$$\begin{aligned}
R^q_{ijk} &= g^{q\lambda}(h_{jk}h_{\lambda i} - h_{ik}h_{\lambda j}) \\
g_{\alpha q}R^q_{ijk} &= g_{\alpha q}g^{q\lambda}(h_{jk}h_{\lambda i} - h_{ik}h_{\lambda j}) \\
&= \delta^\lambda_\alpha(h_{jk}h_{\lambda i} - h_{ik}h_{\lambda j}) \\
&= h_{jk}h_{\alpha i} - h_{ik}h_{\alpha j}
\end{aligned}$$

Now selecting $(i, j, k, \alpha) = (1, 2, 2, 1)$, which we can justify later keeping in mind its similarity with the Riemann curvature tensor:

$$g_{1q}R^q_{122} = h_{22}h_{11} - h^2_{12} = \det h$$

which gives us enough to prove the theorem. Recall:

- Recall the Gauss curvature is *defined* as
$$K = \frac{\det(h)}{\det(g)}$$
- Therefore the Gauss curvature depends on $\det(h)$ and $\det(g)$.
- We have just shown that $\det(h) = g_{1q}R^q_{122}$
- Therefore, $\det(h)$ depends on g and R^q_{122}.

- Recall that the Riemann curvature tensor is defined as:
$$R^q_{ijk} := \frac{\partial \Gamma^q_{jk}}{\partial u^i} - \frac{\partial \Gamma^q_{ik}}{\partial u^j} + \Gamma^l_{jk}\Gamma^q_{il} - \Gamma^l_{ik}\Gamma^q_{jl}.$$
- So we the riemann curvature tensor dependant on the Christoffel symbols Γ^z_{xy} and g.
- Lastly we have the Christoffel symbols *defined intrinsically* by
$$\Gamma^z_{xy} = \frac{1}{2} g^{zw} \left(\frac{\partial g_{wx}}{\partial u^y} + \frac{\partial g_{wy}}{\partial u^x} - \frac{\partial g_{xy}}{\partial u^w} \right)$$

To summarise, We have shown that $\det(h)$ depends on g, and thus K depends on g, completing the proof. □

10. Intuition

We will outline our exposition of this section as follows:

- Show curvature of a curve
- Show signed curvature of a curve
- Show normal curvature definition
- Define the mean curvature at a point p as the average of all normal curvature slices in all directions as
$$H(p) := \frac{1}{2\pi} \int_0^{2\pi} \kappa_\nu(\varphi)\, d\varphi$$
where κ_ν is a function of the angle φ which parametrizes the set of unit vectors in all directions at this point.
- Introduce principal curvatures
- Introduce facts about principal curvature (they are always orthogonal etc.)
- Show that mean curvature is average of principal curvatures.

We denote g_{ij} and h_{ij} by the first and second fundamental forms, respectively. Here we let g be the matrix form of g_{ij}, and A be the matrix form of h_{ij}. As standard notation allows, we denote the principal curvatures by $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$, and define the mean curvature as
$$H = \lambda_1 + \cdots + \lambda_n$$
the squared norm of A is given by $\lambda_1^2 + \cdots + \lambda_n^2$

References

[1] Sigurd B. Angenent. "Shrinking doughnuts". In: *Nonlinear Diffusion Equations and Their Equilibrium States, III (Gregynog, 1989)*. Vol. 7. Progress in Nonlinear Differential Equations and Their Applications. Boston, MA: Birkhäuser Boston, 1992, pp. 21–38.

[2] Tsz-Kiu Aaron Chow, Ka-Wing Chow, and Frederick Tsz-Ho Fong. "Self-Expanders to Inverse Curvature Flows by Homogeneous Functions". In: (2018). arXiv: 1701.03995 [math.DG]. URL: https://arxiv.org/abs/1701.03995.

[3] Gregory Drugan, Frederick Tsz-Ho Fong, and Hojoo Lee. *Rotational symmetry of self-expanders to the inverse mean curvature flow with cylindrical ends.* 2016. arXiv: 1608.02137 [math.DG]. URL: https://arxiv.org/abs/1608.02137.

[4] Gregory Drugan, Hojoo Lee, and Glen Wheeler. "Solitons for the inverse mean curvature flow". In: *Pacific Journal of Mathematics* 284.2 (Aug. 2016), pp. 309–326. ISSN: 0030-8730. DOI: 10.2140/pjm.2016.284.309. URL: http://dx.doi.org/10.2140/pjm.2016.284.309.

[5] Klaus Ecker. *Regularity Theory for Mean Curvature Flow.* Vol. 57. Progress in Nonlinear Differential Equations and their Applications. Boston: Birkhäuser, 2004.

[6] L. C. Evans and J. Spruck. "Motion of Level Sets by Mean Curvature. II". In: *Transactions of the American Mathematical Society* 330.1 (1992), pp. 321–332. ISSN: 00029947. URL: http://www.jstor.org/stable/2154167 (visited on 08/26/2025).

[7] Claus Gerhardt. "Flow of nonconvex hypersurfaces into spheres". In: *Journal of Differential Geometry* 32.1 (1990), pp. 299–314. DOI: 10.4310/jdg/1214445048.

[8] G. Huisken and Australian National University. Centre for Mathematical Analysis. *Flow by Mean Curvature of Convex Surfaces Into Spheres.* Research Report. Centre for Mathematical Analysis, Australian National University. Australian National University, Centre for Mathematical Analysis, 1984. URL: https://books.google.co.nz/books?id=_VxQNAAACAAJ.

[9] Gerhard Huisken and Tom Ilmanen. "The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality". In: *Journal of Differential Geometry* 59 (Nov. 2001). DOI: 10.4310/jdg/1090349447.

[10] Gerhard Huisken and Carlo Sinestrari. "Mean curvature flow with surgeries of two–convex hypersurfaces". In: *Inventiones mathematicae* 175.1 (2009), pp. 137–221. ISSN: 1432-1297. DOI: 10.1007/s00222-008-0148-4. URL: https://doi.org/10.1007/s00222-008-0148-4.

[11] Stephen Kleene and Niels Martin Møller. "Self-shrinkers with a rotational symmetry". In: *Transactions of the American Mathematical Society* 366.8 (Mar. 2014), pp. 3943–3963. ISSN: 1088-6850. DOI: 10.1090/s0002-9947-2014-05721-8. URL: http://dx.doi.org/10.1090/S0002-9947-2014-05721-8.

[12] Kwok-Kun Kwong, Hojoo Lee, and Juncheol Pyo. *Weighted Hsiung-Minkowski formulas and rigidity of umbilical hypersurfaces.* 2016. arXiv: 1608.08955 [math.DG]. URL: https://arxiv.org/abs/1608.08955.

[13] Carlo Mantegazza. *Lecture Notes on Mean Curvature Flow.* Vol. 290. Progress in Mathematics. Basel: Birkhäuser/Springer Basel AG, 2011, pp. xii+166. ISBN: 978-3-0348-0144-7.

[14] John I. E. Urbas. "On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures". In: *Mathematische Zeitschrift* 205.3 (1990). MR 1082861, pp. 355–372. DOI: 10.1007/BF02571249. URL: https://doi.org/10.1007/BF02571249.

[15] Brian White. "Stratification of minimal surfaces, mean curvature flows, and harmonic maps". In: *Journal für die reine und angewandte Mathematik* 488 (1997), pp. 1–36.

[16] Brian White. "The nature of singularities in mean curvature flow of mean-convex sets". In: *Journal of the American Mathematical Society* 16.1 (2003), pp. 123–138.

[17] Brian White. "The size of the singular set in mean curvature flow of mean-convex sets". In: *Journal of the American Mathematical Society* 13.3 (2000), pp. 665–695.

6 Conway Street, Ohakune
Email address: isaacyeshua777@gmail.com