Linear Algebra 2024

1. Basics

1.1 Part One – Regular Questions

Exercise 1.1 Define the matrices

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 2 \\ 3 & 6 & 3 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 3 \\ -4 \\ 9 \end{bmatrix}$$

1. Find the LU decomposition of A *using partial pivoting*.
2. Use the previous result to solve $Ax = b$.

■

To find the LU decomposition of A using partial pivoting I will first move the 3rd row to the top because row 3's pivot has the greatest absolute value. We will then row reduce:

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 2 \\ 3 & 6 & 3 \end{bmatrix} \Longrightarrow_{R_1 \Longleftrightarrow R_3} \begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & 2 \\ 1 & -2 & -3 \end{bmatrix} \Longrightarrow_{R_3 - \frac{1}{3}R_1} \begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & 2 \\ 0 & -4 & -4 \end{bmatrix}$$

We will then need to swap row 2 and 3, and then row reduce:

$$\begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & 2 \\ 0 & -4 & -4 \end{bmatrix} \Longrightarrow_{R_2 \Longleftrightarrow R_3} \begin{bmatrix} 3 & 6 & 3 \\ 0 & -4 & -4 \\ 0 & 3 & 2 \end{bmatrix} \Longrightarrow_{R_3 + \frac{3}{4}R_2} \begin{bmatrix} 3 & 6 & 3 \\ 0 & -4 & -4 \\ 0 & 0 & -1 \end{bmatrix} = U$$

To construct L, we start with the skeleton:

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{2,1} & 1 & 0 \\ a_{3,1} & a_{3,2} & 1 \end{bmatrix}$$

and we have to fill in these remaining entries with the negatives of the coefficients of row reduction that were previously used. Particularly: $a_{3,2} = -\frac{3}{4}$, $a_{3,1} = \frac{1}{3}$, $a_{2,1} = 0$. Obviously we have to swap

$a_{3,1}$ and $a_{2,1}$ because we swap rows in the next step. This gives us

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \end{bmatrix}$$

Now to find the permutation matrix we will see what rows happened to get swapped:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \to \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \to \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \text{whose corresponding P matrix is} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since $LU = PA$ this means $Ax = b$ implies $LUx = Pb$. Letting $y = Ux$ so that $Ly = Pb$, and we can now solve for y.

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 9 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -4 \end{bmatrix}$$

And by back substitution:

$$\begin{aligned} y_1 &= 9 \\ \frac{y_1}{3} + y_2 &= 3 \\ -\frac{3y_2}{4} + y_3 &= -4 \end{aligned}$$

which would yield

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ -4 \end{bmatrix}$$

All that is left is to solve $Ux = y$, which we can do by solving

$$\begin{bmatrix} 3 & 6 & 3 \\ 0 & -4 & -4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ -4 \end{bmatrix}$$

Which by back substitution would (finally) give us:

$$\begin{aligned} & & -x_3 = -4 &\implies & x_3 &= 4 \\ -4x_2 - 4x_3 = 0 &\implies & -4x_2 - 4(4) = 0 &\implies & x_2 &= -4 \\ 3x_1 + 6x_2 + 3x_3 = 9 &\implies & 3x_1 - 24 + 12 = 9 &\implies & x_1 &= \frac{9+12}{3} = 7 \end{aligned}$$

That is:

$$x = \begin{bmatrix} 7 \\ -4 \\ 4 \end{bmatrix}$$

Exercise 1.2 Let

$$Q = \begin{bmatrix} 1 & 2 & k \\ 0 & -1 & 2 \\ k & 2 & 1 \end{bmatrix}$$

Find $\det(Q)$ and use this result to find all values of $k \in \mathbb{R}$ for which the columns of Q do not form a basis for $\mathbb{R}^3$ ■

So all I have to so is to solve $\det(Q)$ which should be

$$\det(Q) = 1(-1-4) - 2(0-2k) + k(0--k) = -5 + 4k + k^2$$

re-arranging I get:

$$k^2 + 4k - 5 = (k+5)(k-1) = 0$$

which means the values of k which would not form a basis for $\mathbb{R}^3$ are $k = -5$ and $k = 1$

Exercise 1.3 Let

$$B = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

1. Prove that B is a basis of $\mathbb{R}^3$. You may do this by computing $\det([B])$, row reducing $[B]$ or other means.
2. Find the change of basis matrix from E to B.
3. Determine $[x]_B$ for the vector $x = [5,4,1]^\top$
4. Multiply the answer above by $[B]$ and show this recovers the vector x.

■

Proof. To show that B is a basis of $\mathbb{R}^3$ we need to show that the vectors that compose B are linearly independent. Or

$$\nexists a,b,c \in \mathbb{R}\backslash\{0\} \quad \text{such that} \quad a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to row reducing

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -2 & -1 \end{bmatrix} \Longrightarrow_{R_3+R_1} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \Longrightarrow_{R_3+R_2} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

which by back substitution gives

$$\begin{aligned} 2c &= 0 && \Longrightarrow c = 0 \\ b + c &= 0 && \Longrightarrow b = 0 \\ -a + b - 2c &= 0 && \Longrightarrow a = 0 \end{aligned}$$

Which thus means that no $v \in B$ can be made by a linear combination of the other vectors unless $a,b,c = 0$ (in that case isn't really a linear combination). ■

Proof. Finding the change of basis matrix from B to the basis $E = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is simply a matter of finding where the standard basis vectors end up after changing basis. This is the same as a linear transformation, whose matrix is composed of the components of B. Or symbolically:,

$$[B] = P_{B\to E} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -2 & -1 \end{bmatrix}$$

■

To find $P_{E\to B}$ we need to find the inverse of the above matrix. We can do this by reducing the augmented matrix $[B|I]$ to $[I,C]$ and this will ensure $C = B^{-1}$. We can start this by:

$$\begin{bmatrix}-1 & 1 & 2 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 1 & 0\\ 1 & -2 & -1 & 0 & 0 & 1\end{bmatrix} \Longrightarrow_{R_1 \Longleftrightarrow R_3} \begin{bmatrix}1 & -2 & -1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\\ -1 & 1 & 2 & 1 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & -1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\\ -1 & 1 & 2 & 1 & 0 & 0\end{bmatrix} \Longrightarrow_{R_3+R_1} \begin{bmatrix}1 & -2 & -1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & -1 & 1 & 1 & 0 & 1\end{bmatrix} \Longrightarrow_{R_3+R_2} \begin{bmatrix}1 & -2 & -1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 2 & 1 & 1 & 1\end{bmatrix}$$

$$\Longrightarrow_{\frac{1}{2}R_3} \begin{bmatrix}1 & -2 & -1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix} \Longrightarrow_{R_2-R_3} \begin{bmatrix}1 & -2 & -1 & 0 & 0 & 1\\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\Longrightarrow_{R_1+R_3} \begin{bmatrix}1 & -2 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2}\\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix} \Longrightarrow_{R_1+2R_2} \begin{bmatrix}1 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2}\\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

This gives us

$$P_{E\to B} = \begin{bmatrix}-\frac{1}{2} & \frac{3}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

Proof. Finding $[x]_B$ for the vector

$$\begin{bmatrix}5\\4\\1\end{bmatrix}$$

is to find three constants $c_1, c_2, c_3 \in \mathbb{R}$ so that

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\\ 0 & 1 & 1\\ 1 & -2 & -1\end{bmatrix} = \begin{bmatrix}5\\4\\1\end{bmatrix}$$

and by augmenting the matrix we simply have to solve:

$$\begin{bmatrix}-1 & 1 & 2 & 5\\ 0 & 1 & 1 & 4\\ 1 & -2 & -1 & 1\end{bmatrix} \Longrightarrow_{R_1 \Longleftrightarrow R_3} \begin{bmatrix}1 & -2 & -1 & 1\\ 0 & 1 & 1 & 4\\ -1 & 1 & 2 & 5\end{bmatrix} \Longrightarrow_{R_3+R_1} \begin{bmatrix}1 & -2 & -1 & 1\\ 0 & 1 & 1 & 4\\ 0 & -1 & 1 & 6\end{bmatrix}$$

$$\Longrightarrow_{R_3+R_2} \begin{bmatrix}1 & -2 & -1 & 1\\ 0 & 1 & 1 & 4\\ 0 & 0 & 2 & 10\end{bmatrix} \Longrightarrow_{\frac{1}{2}R_3} \begin{bmatrix}1 & -2 & -1 & 1\\ 0 & 1 & 1 & 4\\ 0 & 0 & 1 & 5\end{bmatrix} \Longrightarrow_{\substack{R_2-R_3\\ R_1+R_3}} \begin{bmatrix}1 & -2 & 0 & 6\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 5\end{bmatrix}$$

$$\Longrightarrow_{R_1+2R_2} \begin{bmatrix}1 & 0 & 0 & 4\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 5\end{bmatrix}$$

Therefore

$$[x]_B = \begin{bmatrix}4\\-1\\5\end{bmatrix}$$

■

R I just realized that this could've been done faster by multiplying $P_{E\to B}$ by the vector in question. But it took me too long to write the above... So here is my working for the shorter way:

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2}+6+\frac{1}{2} \\ -\frac{5}{2}+2-\frac{1}{2} \\ \frac{5}{2}+2+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$$

Proof. For the last question (4), we simply have to compute $[B][x]_B$, or

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -5-1+8 \end{bmatrix}$$

■

Exercise 1.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 6x_2 - 9x_3 \\ 0 \\ 4x_2 - 6x_3 \end{bmatrix}\right)$$

and let

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

1. Write the matrix representation of T relative to the standard basis; call this matrix A.
2. Find a basis for the column space of A.
3. Use the Dimension Theorem, and your result from (2), to determine $\dim(\ker(T))$.
4. Compute the matrix representation of T with respect to B.
5. Suppose $x \in \mathbb{R}^3$ is such that $[x]_B = [0,1,2]^\top$. Use the answer from (4) to evaluate $[T(x)]_B$ with a single calculation.

■

To do (1), we just have to find a matrix A so that

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_2 - 9x_3 \\ 0 \\ 4x_2 - 6x_3 \end{bmatrix}$$

we could form A as the matrix with columns $T(\mathbf{e_1}), T(\mathbf{e_2}), T(\mathbf{e_3})$, but from observation we can see

$$A = \begin{bmatrix} 0 & 6 & -9 \\ 0 & 0 & 0 \\ 0 & 4 & -6 \end{bmatrix}$$

To do (2) we first row reduce A:

$$\begin{bmatrix} 0 & 6 & -9 \\ 0 & 0 & 0 \\ 0 & 4 & -6 \end{bmatrix} \implies_{R_3 - \frac{4}{6}R_1} \begin{bmatrix} 0 & 6 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & -6+\frac{36}{6} \end{bmatrix}$$

we can see that the end result will have pivots in columns 2 and 3. This makes the basis

$$\mathscr{B}(A) = \left\{ \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ -6 \end{bmatrix} \right\}$$

For (3) the dimension theorem states that for any 3×3 matrix we have $\text{rank}(A) + \text{nullity}(A) = 3$. The rank of a matrix is the number of linearly independent columns in A, which in this particular case is 2. Note that nullity(A) = dim(ker(T)). Therefore, we have dim(ker(T)) = 1.

Now to solve (4), we first need to make find B and B^{-1} in matrix form. We can see that

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

in order to find B^{-1} we have to row reduce the augmented matrix from $[B|I]$ to $[I|C]$ and we can see that $C = B^{-1}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow_{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and finally by $R_2 - R_3$ we get

$$[I|C] = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, to calculate the T with respect to B we need to find $B^{-1}AB$ which is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & -9 \\ 0 & 0 & 0 \\ 0 & 4 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & -3 \\ 0 & 0 & 0 \\ 0 & 4 & -2 \end{bmatrix}$$

which (at last) results in the matrix

$$[T(x)]_B = \begin{bmatrix} (0+0+0) & (6+0+0) & (-3+0+0) \\ (0+0+0) & (0+0-4) & (0+0+2) \\ (0+0+0) & (0+0+4) & (0+0-2) \end{bmatrix} = \begin{bmatrix} 0 & 6 & -3 \\ 0 & -4 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

Finishing off (5) requires calculating

$$\begin{bmatrix} 0 & 6 & -3 \\ 0 & -4 & 2 \\ 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+6-6 \\ 0-4+4 \\ 0+4-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 1.5 Let $T : V \to W$ be a linear transformation with $\ker(T) = \{0\}$ (that is, the kernel of T is the set containing only the zero vector). Suppose $u_1, u_2 \in V$ are linearly independent. Prove that $T(u_1)$ and $T(u_2)$ are linearly independent vectors in W. ■

Proof. Consider

$$c_1 T(u_1) + c_2 T(u_2) = 0$$

we need to show that the only solution is $c_1 = c_2 = 0$. By properties of linear transformations, we can deduce that

$$c_1 T(u_1) + c_2 T(u_2) = T(c_1 u_1) + T(c_2 u_2) = T(c_1 u_1 + c_2 u_2)$$

which gives us

$$T(c_1 u_1 + c_2 u_2) = 0$$

Given that $\ker(T) = \{0\}$, the only vector in V that maps to 0 under T is the zero vector. Therefore:

$$T(c_1u_1 + c_2u_2) = 0 \quad \Longrightarrow \quad c_1u_1 + c_2u_2 = 0$$

By supposition, u_1 and u_2 are linearly independent, the equation $c_1u_1 + c_2u_2 = 0$ implies that $c_1 = 0$ and $c_2 = 0$. Therefore $T(u_1)$ and $T(u_2)$ are linearly independent vectors in W. This shows that linear independence is preserved under linear transformations with trivial kernel.

■

Exercise 1.6 For this question use the standard inner product on $\mathbb{C}^n$ (with $n = 2$). Let

$$u = \begin{bmatrix} i \\ 2i \end{bmatrix}, v = \begin{bmatrix} 2-i \\ 1+i \end{bmatrix}, w = \begin{bmatrix} z \\ 1-i \end{bmatrix}$$

1. Then compute $\|u\|$ and $\|v\|$.
2. Find $z \in \mathbb{C}$ so that w is orthogonal to v.

To solve (1), we can see that the standard inner product is defined as

$$\langle u, v\rangle = u^\star v = \sum_{i=1}^{n} u_i^\star v_i$$

where $u^\star$ is the complex conjugate of u:

$$(a+bi)^\star = a - bi$$

We can use this inner product to define a norm, by $\|x\| = \sqrt{\langle x, x\rangle}$. Because we are working where $n = 2$, we have

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^\star x_1 + x_2^\star x_2}$$

Therefore

$$\|u\| = \sqrt{(-i)(i) + (-2i)(2i)} = \sqrt{1 - 4i^2} = \sqrt{5}$$

and calculating likewise for v we get

$$\|v\| = \sqrt{(2+i)(2-i) + (1+i)(1-i)} = \sqrt{(4-i^2) + (1-i^2)} = \sqrt{4+1+1+1} = \sqrt{7}$$

To solve (2), we need to calculate the inner norm $\langle w, v\rangle$ while keeping the variable z. We will set this norm to 0, to solve for the z. See that

$$\begin{aligned} \langle w, v\rangle = z^\star(2-i) + (1-i)^\star(1+i) &= z^\star(2-i)(1+i)(1+i) \\ &= z^\star(2-i) + 2i \end{aligned}$$

Setting this to zero gives us:

$$\begin{aligned} z^\star(2-i) + 2i &= 0 \\ z^\star &= \frac{-2i}{2-i} \end{aligned}$$

and by taking the complex conjugate of both sides we get

$$\begin{aligned} z^\star &= \frac{-2i}{2-i} \\ z &= \frac{-2i}{2-i} \cdot \frac{2+i}{2+i} = \frac{-2i(2+i)}{(2-i)(2+i)} = \frac{-4i+2}{4+1} = \frac{2}{5} - \frac{4}{5}i \end{aligned}$$

1.2 Part Two - Octave/Matlab Questions

Exercise 1.7 1. Construct a 1x4 vector x comprised of integers in the interval $[-10,10]$

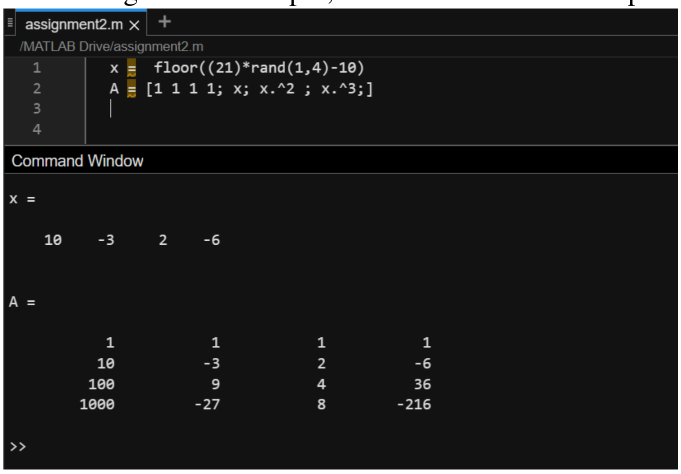
2. Construct the matrix

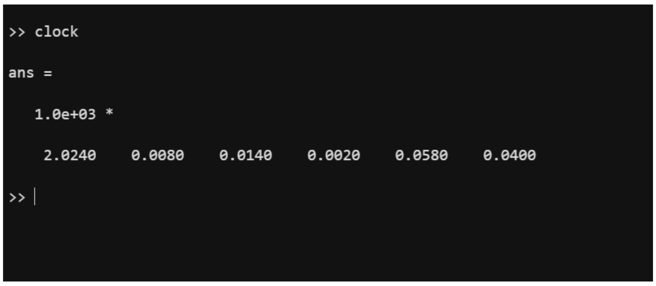
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{bmatrix}$$

where the row $[x_1, x_2, x_3, x_4]$ is the vector obtained in (1). Be as efficient as possible with the construction.

```
% For (1), we utilize the code template from the notes:

x =  floor((21)*rand(1,4)-10)

% For (2), we can set x as the vector above, and the matrix $A$ which we
% want is just the components of the vector squared and cubed.
% For this we can use '.' for component operations:

A = [1 1 1 1; x; x.^2 ; x.^3]

```

The first figure is the output, and the second is the output for clock.





Exercise 1.8 Let

$$A = \begin{bmatrix} 4 & 3 & 1 & 2 & 5 \\ 5 & 5 & 4 & 5 & 5 \\ 6 & 7 & 7 & 8 & 5 \end{bmatrix}$$

- Use the **rref** to evaluate the reduced lower echelon form of A.

Use the previous result to do the following:

1. What is the rank of A?
2. What is the nullity of A?
3. Find a set of columns of A that form a basis for the column space of A.
4. Find the basis for the row space of A.

To solve $(\cdot)$ we can just use **rref** of A. This gives the following output:

```
assignment2.m ×  +
/MATLAB Drive/assignment2.m
A = [
    4 3 1 2 5;
    5 5 4 5 5;
    6 7 7 8 5
];

RA = rref(A)
```

```
Command Window

RA =

    1.0000         0   -1.4000   -1.0000    2.0000
         0    1.0000    2.2000    2.0000   -1.0000
         0         0         0         0         0

>>
```

```
>> clock

ans =

   1.0e+03 *

    2.0240    0.0080    0.0140    0.0030    0.0130    0.0526

>>
```

For (1), we recall that the rank of the matrix is the number of linearly independent columns in A. If we **rref**(A) we can see how many rows are linearly independent by how many pivots there are. In this case there are 2. Another way to do this would be to see how many non zero rows are in **rref**(A).

For (2), we can use the dimension theorem that states that for a 3×5 matrix, the rank plus nullity of A is 4. Since the rank is 2, the nullity is 3.

Moving on to (3). Take the columns with pivots in **rref**(A). The corresponding columns in A, form the basis for the column space of A. From inspection this is

$$B = \left\{ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right\}$$

Lastly for (5), the basis for the row space are the rows in **rref**(A) that have pivots. This is unlike (4) where we had to get the corresponding vectors in the original matrix. We can see this would be the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -\frac{7}{5} \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{11}{5} \\ 2 \\ -1 \end{bmatrix} \right\}$$

Exercise 1.9 A matrix A is said to be *similar* if $A^{\top} = A$ Write an m-file (function) called **symm.m** that inputs a square matrix and has the following features.

1. The function includes a check to verify that the inputted matrix is actually square.
2. The function outputs 1 if the matrix is symmetric, and 0 otherwise.
3. The function provides an appropriate help message.
4. Test your function on the following matrices and show the results:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 \\ 4 & 5 \end{bmatrix}$$

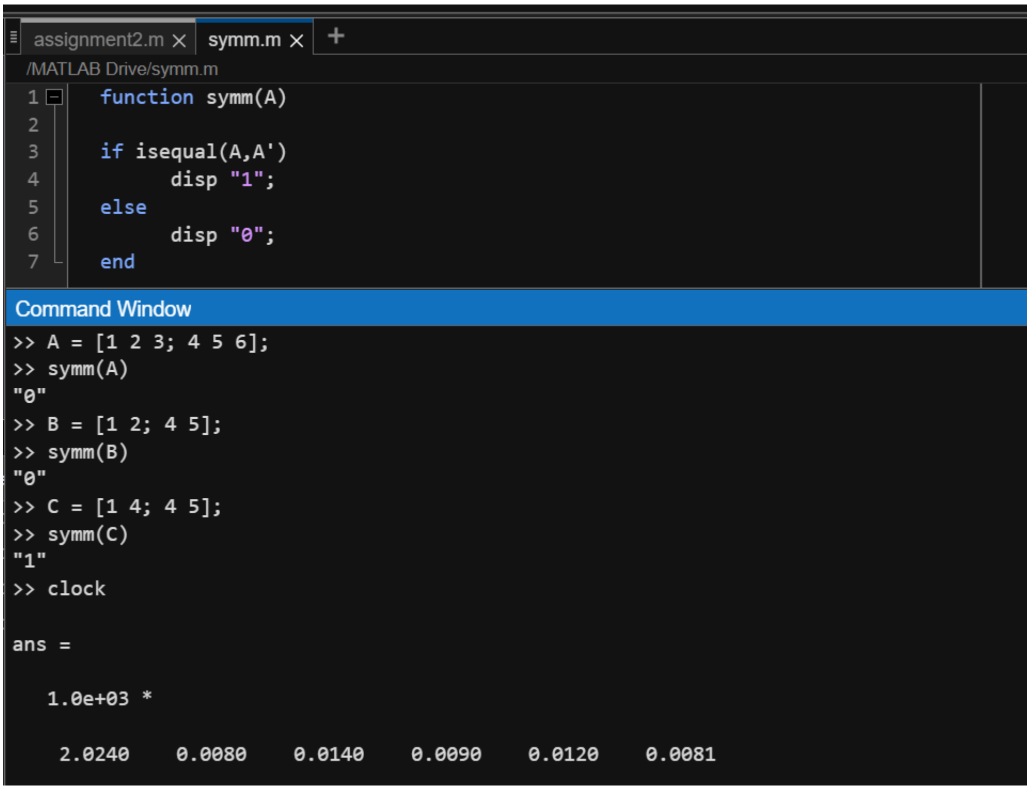
■

R The following is what I tried at first, but not only did it not work, but there were many errors for some reason.

```
function F = symm(A)

if A == A'
      display "1"
else
      display "0"
end
```

The squiggly lines underneath said that using $F = \mathbf{symm}(A)$ is redundant, that $==$ isn't useful for some reason, and display isn't even a command, but rather I had to use disp. Unfortunately, I couldn't come up with something similar to what was in the notes … but the following is the best *working* function I could come up with.

```
function symm(A)
% this function will tell if a matrix is similar or not
% by checking if A is equal to its transpose.
% If it is, then it will return 1, otherwise, it
% will return 0.
if isequal(A,A')
      disp "1";
else
      disp "0";
end
```



2. Level 2

2.1 Part One - Regular Questions

Exercise 2.1 Let W be the subspace of $\mathbb{R}^3$ generated by

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{bmatrix} \right\}$$

1. Find an orthogonal basis for W.
2. Find an orthogonal basis for $W^\perp$.
3. Determine the shortest distance from $x = \begin{bmatrix} 7 & 6 & -7 & -6 \end{bmatrix}^\top$ to W.
4. Find the closest vector in W to the vector x in part (3).

■

Solution for (1). Let u_1 and u_2 be the vectors in the given basis. Let v_1, v_2 be the vectors in the orthogonal basis we are trying to find. First set $v_1 = u_1$.

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

We simply utilise the Gram-Shmidt process. First calculate $u_2 \cdot v_1$ and $v_2 \cdot v_2$ by doing:

$$u_2 \cdot v_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{bmatrix} = 0 + 1 + 12 + 1 = 14$$

$$v_1 \cdot v_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} = 0+1+9+1 = 11$$

To finish finding v_2, we use the formula

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

to get

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{bmatrix} - \frac{14}{11} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-0 \\ 11/11 - 14/11 \\ 44/11 - 42/11 \\ 11/11 - 14/11 \end{bmatrix} = \begin{bmatrix} 0 \\ -3/11 \\ 2/11 \\ -3/11 \end{bmatrix}$$

its easily seen that $v_1 \cdot v_2 = 0$ and thus are orthogonal. So the orthogonal basis is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3/11 \\ 2/11 \\ -3/11 \end{bmatrix} \right\}$$

R I noticed there was a small mention that the Gram - Schmidt process was covered in 160.101? I'm not sure of the other students but I'm pretty sure I don't remember it being covered in my notes either.

For (2), we need to find the orthogonal compliment for W, or

$$\{\mathbf{u} \in \mathbb{R}^4 : \mathbf{u} \cdot \mathbf{v_1} = \mathbf{u} \cdot \mathbf{v_2} = 0\}$$

If we let $A = [u_1, u_2]$ where u_1, u_2 are the vectors in out basis for W, then we can see that $W = CS(A)$... We can see that $W^\perp = NS(A^\top)$. We need to find the null space of

$$A^\top = \begin{bmatrix} 0 & 1 & 3 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix} \xRightarrow{R_2 - R_1} \begin{bmatrix} 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xRightarrow{R_1 - 3R_2} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Which by setting the free variables $x_1 = s$ and $x_4 = t$ we can observe

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} s + \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{v_2} t$$

Since these vectors are already orthogonal ($v_1 \cdot v_2 = 0$), and thus form an orthogonal basis for $W^\perp$. Problem (3) only requires we find the projection of x onto the subspace W. Compute the following:

$$\mathbf{proj}_W(x) = \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2$$

Here x is the vector we are trying to project, and

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ -3/11 \\ 2/11 \\ -3/11 \end{bmatrix}$$

Calculating the individual components:

$$x\cdot v_1=\begin{bmatrix}7\\6\\-7\\-6\end{bmatrix}\cdot\begin{bmatrix}0\\1\\3\\1\end{bmatrix}=6-21-6=-21 \qquad x\cdot v_2=\begin{bmatrix}7\\6\\-7\\-6\end{bmatrix}\cdot\begin{bmatrix}0\\-3/11\\2/11\\-3/11\end{bmatrix}=-\frac{18}{11}-\frac{14}{11}+\frac{18}{11}=-\frac{14}{11}$$

$$v_1\cdot v_1=\begin{bmatrix}0\\1\\3\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\3\\1\end{bmatrix}=1+9+1=11 \qquad v_2\cdot v_2=\begin{bmatrix}0\\-3/11\\2/11\\-3/11\end{bmatrix}\cdot\begin{bmatrix}0\\-3/11\\2/11\\-3/11\end{bmatrix}=\frac{9}{121}+\frac{4}{121}+\frac{9}{121}=\frac{22}{121}$$

noticing that $-14/11*22/121=-7$ and putting it together we have

$$\mathbf{proj}_W(x)=\frac{-21}{11}\begin{bmatrix}0\\1\\3\\1\end{bmatrix}-7\begin{bmatrix}0\\-3/11\\2/11\\-3/11\end{bmatrix}=\begin{bmatrix}0\\-21/11+21/11\\-63/11-14/11\\-21/11+21/11\end{bmatrix}=\begin{bmatrix}0\\0\\-77/11\\0\end{bmatrix}$$

Now to find the distance from the vector $\mathbf{x}$ to the subspace we need to subtract the projection we calculated from $\mathbf{x}$:

$$\begin{bmatrix}7\\6\\-7\\-6\end{bmatrix}-\begin{bmatrix}0\\0\\-7\\0\end{bmatrix}=\begin{bmatrix}7\\6\\0\\-6\end{bmatrix}$$

The length of this is

$$\sqrt{49+36+36}=\sqrt{121}=11$$

Exercise 2.2 A two dimensional linear transformation $T(x)=Mx$ is a *reflection* if M is orthogonal and $\det(M)=-1$. Find all values of $\alpha,\beta\in\mathbb{R}$ so that

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}\alpha x+\frac{5}{13}y\\\beta y+\frac{5}{13}x\end{bmatrix}$$

is a reflection. ■

Well the first step would be to find M, which from observation is

$$M=\begin{bmatrix}\alpha & 5/13\\5/13 & \beta\end{bmatrix}$$

We need to find the real values α and the above β so that M is orthogonal, or

$$M^\top M=\begin{bmatrix}\alpha & 5/13\\5/13 & \beta\end{bmatrix}\begin{bmatrix}\alpha & 5/13\\5/13 & \beta\end{bmatrix}=\begin{bmatrix}\alpha^2+25/169 & \frac{5\alpha+5\beta}{13}\\\frac{5\alpha+5\beta}{13} & \beta^2+25/169\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Something useful from the above is that

$$\frac{5}{13}(\alpha+\beta)=0 \quad\Longrightarrow\quad \beta=-\alpha$$

In effect, this turns our original matrix into

$$M=\begin{bmatrix}\alpha & 5/13\\5/13 & -\alpha\end{bmatrix}$$

notice that $\det(M)$ is $-\alpha^2 - 25/169$. Setting this to -1 and solving for α is a simple matter:

$$\begin{aligned} -\alpha^2 - \frac{25}{169} &= -1 \\ -\alpha &= \frac{25}{169} - \frac{169}{169} \\ \alpha^2 &= \frac{144}{169} \end{aligned}$$

So this gives us the two values $12/13$ and $-12/13$ for α. Since β is just the negative of α we get our answer to be both

$$\left(\frac{12}{13}, -\frac{12}{13}\right) \quad \text{and} \quad \left(-\frac{12}{13}, \frac{12}{13}\right)$$

R I'm not sure why information on the determinant of M was necessary. From M being orthogonal we can see that not only is $\beta = -\alpha$, but $\alpha^2 = 1 - 25/169$. Solving this would give the same result. I chose to use det just in case I'm missing something.

Exercise 2.3 Find a sequence of two planar rotations that transform $x = \begin{bmatrix} 6 & 6 & 7 \end{bmatrix}^\top$ to a multiple of $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$ ■

We will first rotate using the first and second coordinates $x_1 = 6$ and $x_2 = 6$. Our first rotation matrix is

$$\begin{bmatrix} 6/\sqrt{6^2+6^2} & 6/\sqrt{6^2+6^2} & 0 \\ -6/\sqrt{6^2+6^2} & 6/\sqrt{6^2+6^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6/6\sqrt{2} & 6/6\sqrt{2} & 0 \\ -6/6\sqrt{2} & 6/6\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

clearly the above matrix multiplied by x gives

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 6/\sqrt{2} + 6/\sqrt{2} \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 12/\sqrt{2} \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 6\sqrt{2} \\ 0 \\ 7 \end{bmatrix}$$

which indeed gets rid of the second coordinate. To get rid of the third coordinate, we find the rotation matrix with coordinates $x_1 = 6\sqrt{2}$ and $x_3 = 7$. First note that $\sqrt{x_1^2 + x_2^2} = \sqrt{(6\sqrt{2})^2 + 49} = \sqrt{72 + 49} = \sqrt{121} = 11$.

$$\begin{bmatrix} 6\sqrt{2}/11 & 0 & 7/11 \\ 0 & 1 & 0 \\ -7/11 & 0 & 6\sqrt{2}/11 \end{bmatrix}$$

Notice that multiplying the above by $\begin{bmatrix} 12/\sqrt{2} & 0 & 7 \end{bmatrix}^\top$ gives

$$\begin{bmatrix} 6\sqrt{2}/11 & 0 & 7/11 \\ 0 & 1 & 0 \\ -7/11 & 0 & 6\sqrt{2}/11 \end{bmatrix} \begin{bmatrix} 6\sqrt{2} \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 36 \cdot 2/11 + 49/11 \\ 0 \\ -42/11\sqrt{2} + 42\sqrt{2}/11 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix}$$

which is clearly a multiple of the first unit vector $\mathbf{e_1} \in \mathbb{R}^3$. To check our final answer, we know that $|x| = |11\mathbf{e_1}|$ has to hold. Since x has magnitude $\sqrt{36+36+49} = 11$ and so does our answer, we can conclude.

Exercise 2.4 Let

$$A = \begin{bmatrix} 8 & 8 & 3 \\ 4 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

1. Perform two Householder transformations to transform A into an upper triangular matrix R.
2. Use your Householder transformations from (1) to identity an orthogonal matrix Q such that $A = QR$.
3. Use your result from (2) to evaluate $\det(A)$.
4. Use your result from (2) to solve $Ax = b$

■

To solve (1) we first let $u_1 = \begin{bmatrix} 8 & 4 & 1 \end{bmatrix}^\top$, which has norm $\sqrt{64+16+1} = \sqrt{81} = 9$. This means we can set $v_1 = \begin{bmatrix} 9 & 0 & 0 \end{bmatrix}^\top$, and $w_1 = u_1 - v_1 = \begin{bmatrix} -1 & 4 & 1 \end{bmatrix}^\top$. Computing the first householder transformation gives

$$H_1 = I - \frac{2}{\|w_1\|^2} w_1 w_1^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{18} \begin{bmatrix} -1\cdot -1 & -1\cdot 4 & -1\cdot 1 \\ 4\cdot -1 & 4\cdot 4 & 4\cdot 1 \\ 1\cdot -1 & 1\cdot 4 & 1\cdot 1 \end{bmatrix} =$$

$$\frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & -4 & -1 \\ -4 & 16 & 4 \\ -1 & 4 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} -1 & 4 & 1 \\ 4 & -16 & -4 \\ 1 & -4 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 8 & 4 & 1 \\ 4 & -7 & -4 \\ 1 & -4 & 8 \end{bmatrix}$$

notice that

$$H_1 A = \frac{1}{9}\begin{bmatrix} 8 & 4 & 1 \\ 4 & -7 & -4 \\ 1 & -4 & 8 \end{bmatrix} \begin{bmatrix} 8 & 8 & 3 \\ 4 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 64+16+1 & 64-4+3 & 24+4-1 \\ 32-28-4 & 32+7-12 & 12-7+4 \\ 8-16+8 & 8+4+24 & 3-4-8 \end{bmatrix} =$$

$$\frac{1}{9}\begin{bmatrix} 81 & 63 & 27 \\ 0 & 27 & 9 \\ 0 & 36 & -9 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 3 \\ 0 & 3 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

We have our new matrix as $\left[\begin{smallmatrix} 7 & 3 \\ 3 & 1 \\ 4 & -1 \end{smallmatrix}\right]$. This time let $u_2 = \left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right]$, which has norm $\sqrt{9+16} = 5$. Let $v_2 = \left[\begin{smallmatrix} 5 \\ 0 \end{smallmatrix}\right]$ and set $w_2 = u_2 - v_2 = \left[\begin{smallmatrix} -2 \\ 4 \end{smallmatrix}\right]$. We first calculate K_2;

$$K_2 = I - \frac{2}{\|w_1\|^2} w_1 w_1^\top = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{20}\begin{bmatrix} -2\cdot -2 & -2\cdot 4 \\ 4\cdot -2 & 4\cdot 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 4 & -8 \\ -8 & 16 \end{bmatrix} =$$

$$\frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

We embed this into a 3×3 identity matrix, keeping 5 as the (1,1) entry.

$$H_2 = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

We multiply this by the previous transformation:

$$H_2 H_1 A = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix} \begin{bmatrix} 9 & 7 & 3 \\ 0 & 3 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

Which would give

$$R = H_2H_1A = \frac{1}{5}\begin{bmatrix}5 & 35 & 15\\ 0 & 9+16 & 3-4\\ 0 & 12-12 & 4+3\end{bmatrix} = \frac{1}{5}\begin{bmatrix}5 & 35 & 15\\ 0 & 25 & -1\\ 0 & 0 & 7\end{bmatrix} = \begin{bmatrix}1 & 7 & 3\\ 0 & 5 & -1/5\\ 0 & 0 & 7/5\end{bmatrix}$$

For (2), we simply calculate

$$Q = H_1H_2 = \frac{1}{9}\begin{bmatrix}8 & 4 & 1\\ 4 & -7 & -4\\ 1 & -4 & 8\end{bmatrix}\frac{1}{5}\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 4\\ 0 & 4 & -3\end{bmatrix} = \frac{1}{45}\begin{bmatrix}40 & 12+4 & 16-3\\ 20 & -21-16 & -28+12\\ 5 & -12+32 & -16-24\end{bmatrix}$$

$$= \frac{1}{45}\begin{bmatrix}40 & 16 & 13\\ 20 & -37 & -16\\ 5 & 20 & -40\end{bmatrix}$$

R To check if the Q given above is orthogonal calculate $QQ^\top$ and check if it equal to I.

$$\frac{1}{45^2}\begin{bmatrix}40 & 16 & 13\\ 20 & -37 & -16\\ 5 & 20 & -40\end{bmatrix}\begin{bmatrix}40 & 20 & 5\\ 16 & -37 & 20\\ 13 & -16 & -40\end{bmatrix}$$

$$= \frac{1}{2025}\begin{bmatrix}1600+256+169 & 800-592-208 & 200+320-520\\ 800-592-208 & 400+1369+256 & 100-740+640\\ 200+320-520 & 100-740+640 & 25+400+1600\end{bmatrix}$$

$$= \frac{1}{2025}\begin{bmatrix}2025 & 0 & 0\\ 0 & 2025 & 0\\ 0 & 0 & 2025\end{bmatrix} = I$$

Unfortunately it is a bit difficult to calculate the above equation in my head, so I used a calculator for this.

We can use this Q to solve (3). First note $\det(R) = 1\cdot 7\cdot 7/5 = 7$. Recall the peculiar property $\det(AB) = \det(A)\cdot\det(B)$. Given $A = QR$ we can deduce that $\det(A) = \det(Q)\det(A)$. Notice that in general settings $Q = H_1H_2\cdots H_{n-1}$, where n is the dimension of the matrix A. We can use the fact that all householder transformations have determinant -1 to get the following formula:

$$\det(A) = \det(R)(-1)^{n-1}$$

Alternatively:

$$\det(A) = \det(QR) = \det(Q)\det(R) = \det(H_1H_2)\det(R) = \det(H_1)\det(H_2)\det(R)$$

which would indeed give the same answer 7. Lastly, for (4) we just need to compute $Rx = Q^\top b$:

$$\begin{bmatrix}1 & 7 & 3\\ 0 & 5 & -1/5\\ 0 & 0 & 7/5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \frac{1}{45}\begin{bmatrix}40 & 20 & 5\\ 16 & -37 & 20\\ 13 & -16 & -40\end{bmatrix}\begin{bmatrix}1\\ -3\\ 1\end{bmatrix}$$

The second half of this equation gives:

$$\frac{1}{45}\begin{bmatrix}40 & 20 & 5\\ 16 & -37 & 20\\ 13 & -16 & -40\end{bmatrix}\begin{bmatrix}1\\ -3\\ 1\end{bmatrix} = \frac{1}{45}\begin{bmatrix}40-60+5\\ 16+111+20\\ 13+48-40\end{bmatrix} = \frac{1}{45}\begin{bmatrix}-15\\ 147\\ 21\end{bmatrix}$$

which, in turn, yields the set of simultaneous equations

$$\begin{aligned}
& & \frac{7}{5}x_3 = \frac{21}{45} \quad & \implies x_3 = \frac{5(21)}{45(7)} = \frac{21}{63} = \frac{1}{3} \\
5x_2 - x_3/5 = 147/45 \quad & \implies 5x_2 - 1/15 = 147/45 \quad & & \implies x_2 = (147/45 + 3/45)/5 \\
& & & = \frac{150}{225} = \frac{2}{3} \\
x_1 + 7x_2 + 3x_3 = \frac{1}{3} \quad & \implies x_1 + \frac{14}{3} + 1 = -\frac{1}{3} \quad & & \implies x_1 = -\frac{15}{3} - 1 = -6
\end{aligned}$$

So our final answer is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 2/3 \\ 1/3 \end{bmatrix}$$

It is clear (with hard work) that $Ax = b$ is indeed true.

Exercise 2.5 Find the least squares curve of best fit to the data in the table below, by a quadratic of the form $y = ax + bx^2$.

x	y
-2	8
-1	7
1	-6
3	1

■

We start by deriving

$$A = \begin{bmatrix} -2 & 4 \\ -1 & 1 \\ 1 & 1 \\ 3 & 9 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 8 \\ 7 \\ -6 \\ 1 \end{bmatrix}$$

the constants a and b we are looking for will be the entries of vector which is the result of $(A^\top A)^{-1}A^\top b$. Lets calculate these individually

$$(A^\top A)^{-1} = \left(\begin{bmatrix} -2 & -1 & 1 & 3 \\ 4 & 1 & 1 & 9 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ -1 & 1 \\ 1 & 1 \\ 3 & 9 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 4+1+1+9 & -8-1+1+27 \\ -8-1+1+27 & 16+1+1+81 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} 15 & 19 \\ 19 & 99 \end{bmatrix}^{-1} = \frac{1}{1487-361} \begin{bmatrix} 99 & -19 \\ -19 & 15 \end{bmatrix} = \frac{1}{1124} \begin{bmatrix} 99 & -19 \\ -19 & 15 \end{bmatrix}$$

$$A^\top b = \begin{bmatrix} -2 & -1 & 1 & 3 \\ 4 & 1 & 1 & 9 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} -16-7-6+3 \\ 32+7-6+9 \end{bmatrix} = \begin{bmatrix} -26 \\ 42 \end{bmatrix}$$

The final product:

$$(A^\top A)^{-1}A^\top b = \frac{1}{1124} \begin{bmatrix} 99 & -19 \\ -19 & 15 \end{bmatrix} \begin{bmatrix} -26 \\ 42 \end{bmatrix} = \frac{1}{1124} \begin{bmatrix} -2574-798 \\ 494+630 \end{bmatrix} = \frac{1}{1124} \begin{bmatrix} -3372 \\ 1124 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

This means our final formula is $y = x^2 - 3x$.

Exercise 2.6 Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Find all right inverses of A. ■

Essentially, we need to solve for $A^- = A^\top(AA^\top)^{-1}$, and find the null space Z of $A^\top$, then any right inverse can be expressed as $A^- + Z$. Notice that

$$(AA^\top)^{-1} = \left(\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \right) = \begin{bmatrix} 5 & 5 \\ 5 & 6 \end{bmatrix}^{-1} = \frac{1}{30-25} \begin{bmatrix} 6 & -5 \\ -5 & 5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & -5 \\ -5 & 5 \end{bmatrix}$$

$$A^- = A^\top(AA^\top)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 6/5 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 0 \\ -1 & 1 \\ 2/5 & 0 \end{bmatrix}$$

To find Z, we first row reduce A by subtracting the first column from the second to get $\left(\begin{smallmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. Solving $Ax = 0$ gives us the simultaneous equations: $x_1 = -2x_3$, $x_2 = 0$, $x_3 = x_3$. This indeed gives us

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\text{basis}} x_3$$

Since both A and Z are 2×3 matrices, we get

$$Z = \begin{bmatrix} -2s & -2t \\ 0 & 0 \\ s & t \end{bmatrix}$$

Therefore the final right inverse is

$$A^- + Z = \begin{bmatrix} 1/5 & 0 \\ -1 & 1 \\ 2/5 & 0 \end{bmatrix} + \begin{bmatrix} -2s & -2t \\ 0 & 0 \\ s & t \end{bmatrix} = \underbrace{\begin{bmatrix} 1/5 - 2s & -2t \\ -1 & 1 \\ 2/5 + s & t \end{bmatrix}}_{\text{for all } s,t \in \mathbb{R}}$$

To check this, calculate:

$$A(A^- + Z) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1/5 - 2s & -2t \\ -1 & 1 \\ 2/5 + s & t \end{bmatrix} = \begin{bmatrix} 1/5 - 2s + 4/5 + 2s & -2t + 2t \\ 1/5 - 2s - 1 + 4/5 + 2s & -2t + 1 - 2t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 2.7 Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

1. Find all least squares solutions to $Ax = b$.
2. Use the answer to part (a) to find the minimum-length least squares solution. **Hint**: you may either complete the square or use calculus to find the minimum.
3. Find the Moore-Penrose inverse of A.
4. Use your Moore-Penrose inverse to find the minimum length least squares solution to $Ax = b$ a second time.

To find all least squares solutions to $Ax = b$, we first find $A^\top Ax$ and $A^\top b$, then solve for x. Here

$$A^\top A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+4+1 & -1-4-1 & 2+6+1 \\ -1-4-1 & 1+4+1 & -2-6-1 \\ 2+6+1 & -2-6-1 & 4+9+1 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & -6 & 9 \\ -6 & 6 & -9 \\ 9 & -9 & 14 \end{bmatrix}$$

$$A^\top b = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-2+3 \\ -2+2-3 \\ 4-3+3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

We create the augmented matrix $[A^\top A | A^\top b]$ and row reduce:

$$\begin{bmatrix} 6 & -6 & 9 & 3 \\ -6 & 6 & -9 & -3 \\ 9 & -9 & 14 & 4 \end{bmatrix} \Longrightarrow_{R_2+R_1} \begin{bmatrix} 6 & -6 & 9 & 3 \\ 0 & 0 & 0 & 0 \\ 9 & -9 & 14 & 4 \end{bmatrix} \Longrightarrow_{\substack{\frac{1}{6}R_1 \\ R_2 \Longleftrightarrow R_3}} \begin{bmatrix} 1 & -1 & 3/2 & 1/2 \\ 9 & -9 & 14 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Longrightarrow_{R_2-9R_1} \begin{bmatrix} 1 & -1 & 3/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Multiplying each row by 2 gives the following set of equations:

$$x_3 = -\frac{1}{2}$$
$$x_2 = x_2$$
$$x_1 - x_2 + \frac{3x_3}{2} = \frac{1}{2} \quad \Longrightarrow x_1 = \frac{2}{4} + \frac{3}{4} + x_2 = x_2 + 1$$

It follows

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t \qquad \text{for some } t \in \mathbb{R}$$

For (2), we need to find the minimum value of

$$\|x\|^2 = \left\| \begin{bmatrix} 1+t \\ t \\ -\frac{1}{2} \end{bmatrix} \right\|^2 = (1+t)^2 + t^2 + \frac{1}{4} = 1 + 2t + 2t^2 + \frac{1}{4}$$
$$= \frac{4}{4} + 2t + 2t^2 + \frac{1}{4} = \frac{5}{4} + 2t + 2t^2$$

Finding the derivative of the above and setting it to 0 gives $2 + 4t = 0$ or $t = -1/2$. Now to find the Moore-Penrose inverse for A we utilise the following algorithm and present the computations for each step, in the order given:

```
--> Row reduce A to RREF;
--> Discard any rows of zeros from RREF(A) and set it to C;
--> Let F = CA';
--> Compute A^G = C' inv((FF')) F;

%Note that A' indicates transpose of A and inv(A) is the inverse
```

1.

$$\begin{bmatrix}1 & -1 & 2\\ 2 & -2 & 3\\ 1 & -1 & 1\end{bmatrix} \underset{\substack{R_2-2R_1\\ R_3-R_1}}{\Longrightarrow} \begin{bmatrix}1 & -1 & 2\\ 0 & 0 & -1\\ 0 & 0 & -1\end{bmatrix} \underset{R_3-R_2}{\Longrightarrow} \begin{bmatrix}1 & -1 & 2\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}$$

2.

$$C = \begin{bmatrix}1 & -1 & 2\\ 0 & 0 & -1\end{bmatrix}$$

3.

$$F = \begin{bmatrix}1 & -1 & 2\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ -1 & -2 & -1\\ 2 & 3 & 1\end{bmatrix} = \begin{bmatrix}1+1+4 & 2+2+6 & 1+1+2\\ -2 & -3 & -1\end{bmatrix}$$

$$= \begin{bmatrix}6 & 10 & 4\\ -2 & -3 & -1\end{bmatrix}$$

4.

$$A^G = \begin{bmatrix}1 & 0\\ -1 & 0\\ 2 & -1\end{bmatrix}\left(\begin{bmatrix}6 & 10 & 4\\ -2 & -3 & -1\end{bmatrix}\begin{bmatrix}6 & -2\\ 10 & -3\\ 4 & -1\end{bmatrix}\right)^{-1}\begin{bmatrix}6 & 10 & 4\\ -2 & -3 & -1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ -1 & 0\\ 2 & -1\end{bmatrix}\left(\begin{bmatrix}152 & -46\\ -46 & 14\end{bmatrix}\right)^{-1}\begin{bmatrix}6 & 10 & 4\\ -2 & -3 & -1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ -1 & 0\\ 2 & -1\end{bmatrix}\frac{1}{12}\begin{bmatrix}14 & 46\\ 46 & 152\end{bmatrix}\begin{bmatrix}6 & 10 & 4\\ -2 & -3 & -1\end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix}1 & 0\\ -1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}-8 & 2 & 10\\ -28 & 4 & 32\end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix}-8 & 2 & 10\\ 8 & -2 & -10\\ 12 & 0 & -12\end{bmatrix}$$

$$= \begin{bmatrix}-2/3 & 1/6 & 5/6\\ 2/3 & -1/6 & -5/6\\ 1 & 0 & -1\end{bmatrix}$$

Believe it or not this NOT the correct matrix. I'd like to think I don't give up easily, but this I will give an exception to (this is the 1550 line of LaTeX).

Exercise 2.8 Let

$$A = \begin{bmatrix}1 & c & 0\\ 0 & c & 1\\ 1 & 1 & -1\end{bmatrix}$$

1. Find $c \in \mathbb{R}$ such that $\lambda = 2$ is an eigenvalue of A.
2. Find $c \in \mathbb{R}$ such that $\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$ is an eigenvector for A.

■

To solve (1), first note

$$\det(\lambda I - A) = \det\left(\begin{bmatrix}\lambda - 1 & -c & 0\\ 0 & \lambda - c & -1\\ -1 & -1 & \lambda + 1\end{bmatrix}\right)$$

Since we want $\lambda = 2$, we have

$$\det\left(\begin{bmatrix} 1 & -c & 0 \\ 0 & 2-c & -1 \\ -1 & -1 & 3 \end{bmatrix}\right) = [3(2-c)-1] + c(0-1) = 6-3c-1-c = 5-4c$$

Setting this to 0 we have $5-4c = 0 \implies c = 5/4$. As for (2), recall that x is an eigenvector if for some $\lambda \in \mathbb{R}$ we have $Ax = \lambda x$. This expands to the equation

$$\begin{bmatrix} 1 & c & 0 \\ 0 & c & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}$$

We can see that from the third row we have $\lambda = 1+1-1 = 1$. Therefore $c = 0$ is the only $c \in \mathbb{R}$ that could solve the other equations.

2.2 Part Two - Octave/Matlab Questions

Exercise 2.9 Write an m-file called $htr.m$ that computes a Householder transformation. The function will input a vector u, and output the Householder transformation H that transforms u to a vector $\begin{bmatrix} t & 0 & \cdots & 0 \end{bmatrix}^\top$, for the appropriate value of $t \geq 0$. If the vector u is of the form $\begin{bmatrix} t & 0 & \cdots & 0 \end{bmatrix}^\top$, the function will output the identity matrix of the appropriate size. Test this with the vectors

$$u = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \quad u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

First we show our command for **clock**:

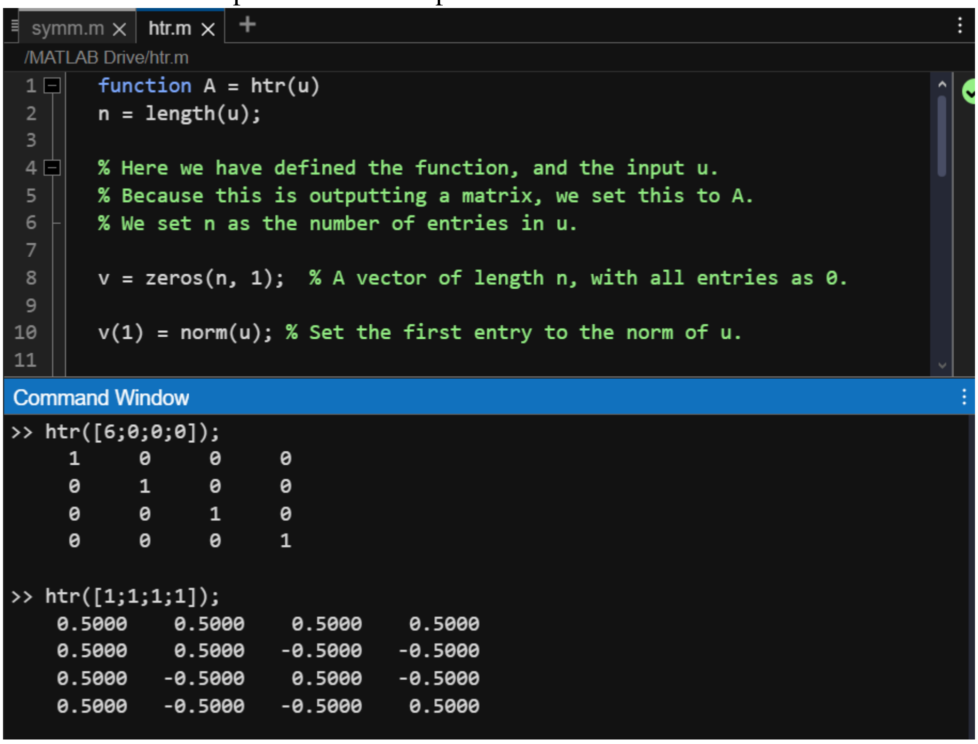
```
Command Window
>> clock

ans =

   1.0e+03 *

    2.0240    0.0090    0.0150    0.0100    0.0060    0.0586

>> |
```

Let's layout our diabolical plan to compute this householder transformation. We will first tell the computer to give us a vector of the same length as u but all of the entries are 0. We will then change the first entry to the norm of u, and set this to v. This is the basis we want to transform to. Set $w = u - v$. If w is 0, then clearly u and v are the same vector, so we output the identity matrix. Otherwise, we calculate H with the formula $H = I - (2 \cdot (w \cdot w))/\|w\|^2$.

```
function A = htr(u)
```

```
n = length(u);

% Here we have defined the function, and the input u.
% Because this is outputting a matrix, we set this to A.
% We set n as the number of entries in u.

v = zeros(n, 1);  % A vector of length n, with all entries as 0.

v(1) = norm(u); % Set the first entry to the norm of u.

w = u-v;

if w ~= zeros(n,1)
    A = eye(n) - (2* (w*w'))/(norm(w)^2);
else
    A = eye(n);
end

% If w not equal to the zero vector, use the formula
% If w is the zero vector, output the identity matrix of dimensions n.

disp(A)
```

Here are the correct outputs for the example vectors:



Exercise 2.10 Fit an exponential equation $y = Ce^{bt}$ to the data:

y	.2	.5	.8	1.1	1.4	1.7	2
t	3.68	2.16	1.18	.73	.35	.33	.10

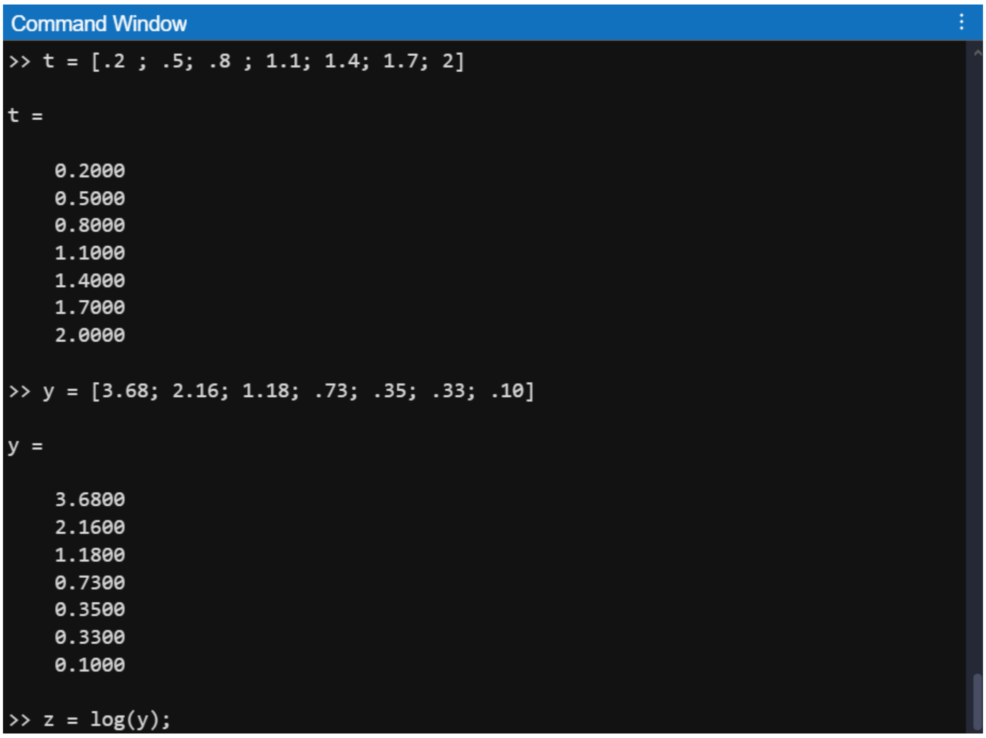
To do this using least squares let $z = \ln(y)$ so that

$$\begin{aligned} z &= \ln(Ce^{bt}) \\ &= \ln(C) = bt \\ &= a + bt \end{aligned}$$

where $a = \ln(C)$.

1. Enter the above data as 7 dimensional vectors t and y into matlab.
2. Use **log** to compute $z = \ln(y)$.
3. Determine the values of a and b that give the least squares solution to $z = a + bt$.
4. Use the command **exp** to evaluate the corresponding value of C.
5. Plot the data and your fitted curve $y = Ce^{bt}$ on the same set of axes.

Let's follow along, and post our results to see what happens. We will first input y and t as vectors. And also set $z = \ln(y)$.

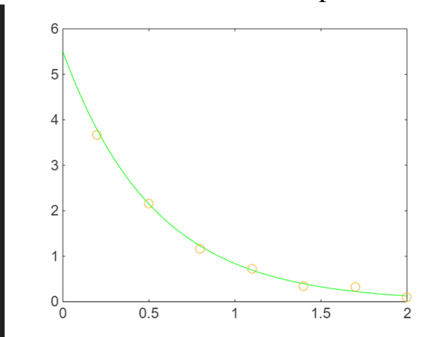


```
Command Window
>> t = [.2 ; .5; .8 ; 1.1; 1.4; 1.7; 2]

t =

    0.2000
    0.5000
    0.8000
    1.1000
    1.4000
    1.7000
    2.0000

>> y = [3.68; 2.16; 1.18; .73; .35; .33; .10]

y =

    3.6800
    2.1600
    1.1800
    0.7300
    0.3500
    0.3300
    0.1000

>> z = log(y);
```

To determine the least squares solution to $z = a + bt$ we need to let A be the matrix with all entries in first column as 1, and all entries in column 2 congruent to t. We then need to solve the equation $(A^\top A)^{-1}(A^\top z)$. We can also calculate the exponent of this result:

```
A =

    1.0000    0.2000
    1.0000    0.5000
    1.0000    0.8000
    1.0000    1.1000
    1.0000    1.4000
    1.0000    1.7000
    1.0000    2.0000

>> (inv(A' * A))*(A' *z)

ans =

    1.7052
   -1.8797

>> exp((inv(A' * A))*(A' *z))

ans =

    5.5024
    0.1526

>>
```

This gives the formula $y = 5.5024e^{-1.8797t}$. This is its plot:



Command Window

```
>> hold on

plot(tVec,5.5024*exp(-1.8797*tVec),'g-')

scatter(t,y)

hold off
```

3. Paint it black

3.1 Part One - Regular Questions

Exercise 3.1 Let

$$A = \begin{bmatrix} -7 & 4 \\ -9 & 8 \end{bmatrix}$$

1. Use the Cayley-Hamilton Theorem to express A^2 and A^3 as linear combinations of I and A.
2. Use the Cayley-Hamilton Theorem to express A^{-1} as a linear combination of I and A.
3. For any positive integer n the matrix A^{-n} is defined to be $(A^{-1})^n$ (provided A is invertible). Use the Cayley-Hamilton Theorem to express A^{-2} as a linear combination of I and A.

■

We first note the characteristic polynomial of A, denoted $q_A(x)$, is calculated by

$$q_A(x) = \det(A - Ix) = \det\left(\begin{bmatrix} -7 & 4 \\ -9 & 8 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\right) = \det\left(\begin{bmatrix} -7-x & 4 \\ -9 & 8-x \end{bmatrix}\right)$$

which is equal to $(-7-x)(8-x) - (-9 \cdot 4) = x^2 - x - 20$. By the lovely Cayley-Hamilton theorem we have $p_A(A) = 0$, and thus

$$A^2 - A - 20I = 0 \tag{3.1}$$

this implies $A^2 = A + 20I$ which answers the first part of question 1. Substituting this answer into the equation $A^3 = AA^2$ shows

$$A^3 = AA^2 = A(A+20I) = A^2 + 20A = A + 20I + 20A = 21A + 20I$$

To solve (2), we first re-arrange equation (3.1) to see $20I = A^2 - A$ and thus

$$I = A\frac{1}{20}(A - I) \quad \Longrightarrow \quad A^{-1} = \frac{1}{20}A - \frac{1}{20}I$$

We have already used the Cayley-Hamilton theorem to find A^{-1}. To find A^{-2} we need only substitute. Here we have

$$A^{-2} = (A^{-1})^2 = \left(\frac{1}{20}(A-I)\right)^2 = \left(\frac{1}{400}(A^2-2A+I)\right)$$

Substituting from our previous result for A^2 we can reduce the above equation to

$$A^{-2} = \frac{1}{400}(A+20I-2A+I) = \frac{1}{400}(-A+21I) = -\frac{1}{400}A + \frac{21}{400}I$$

Exercise 3.2 The land of Zoo Nearland consists of two islands, the East Island and the West Island. Each year 10% of the population of the East Island moves to the West Island, and 20% of the population of the West Island moves to the East Island. Assume that the populations of the islands do not change for any other reason.

1. Let e_k and w_k denote the populations of the East and West Islands, respectively, in millions at the start of the k-th year for each k. Find expressions for e_{k+1} and w_{k+1} in terms of e_k and w_k, and hence find a difference equation $x_{k+1} = Ax_k$ for $x_k = \begin{bmatrix} e_k \\ w_k \end{bmatrix}$
2. Find the solution to the difference equation if there are initially 3 million people on each island.
3. What is the population of each island in the limit as $k \to \infty$?

■

The first step is to model the vector x_{k+1}. We deduce from the information that the components can be expressed as $e_{k+1} = .9e_k + .2w_k$ and $w_{k+1} = .1e_k + .8w_k$. This shows that

$$x_{k+1} = \begin{bmatrix} e_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} .9e_k + .2w_k \\ .1e_k + .8w_k \end{bmatrix} = \underbrace{\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} e_k \\ w_k \end{bmatrix}}_{x_k} = Ax_k$$

The above is our difference equation. To diagonalise this, first find the eigenvalues from the characteristic polynomial

$$\det(A-\lambda I) = \begin{bmatrix} .9-\lambda & .2 \\ .1 & .8-\lambda \end{bmatrix} = (.9-\lambda)(.8-\lambda) - .02 = .72 - 1.7\lambda + \lambda^2 - .02$$

Since $\lambda^2 - 1.7\lambda + .7 = (\lambda-1)(\lambda-.7)$ the eigenvectors are $\lambda_1 = 1, \lambda_2 = .7$. To find the eigenvectors we will solve $(A-\lambda_1 I)v = 0$, and $(A-\lambda_2 I)v = 0$, for some v (or any equivalent scalar multiple of it). See that for λ_1 and λ_2 we have

$$(A-\lambda_1 I)v = \begin{bmatrix} .9-1 & .2 \\ .1 & .8-1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$(A-\lambda_2 I)v = \begin{bmatrix} .9-.7 & .2 \\ .1 & .8-.7 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

From observation we deduce a possible combination of x_1 and y_1 to be 2 and 1 respectively. Using the same argument for λ_2 we can see $x_2 = -1$ and $y_2 = 1$ (or the other way around). Set the matrix P as the columns of the eigenvectors we have found. The diagonal matrix D has our eigenvalues in the diagonal, and zeros elsewhere. We will also need to use P^{-1}.

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & .7 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{\underbrace{\det(P)}_{2-(-1)=3}} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

To get our final answer x_{k+1} we will compute $A^k x_0 = (PD^kP^{-1})x_0$ where x_0 is the 2 dimensional vector with both entries 3.

$$\begin{aligned}
PD^kP^{-1}x_0 &= \begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & .7\end{bmatrix}^k \frac{1}{3}\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3\\ 3\end{bmatrix}\\
&= \begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & (.7)^k\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\\
&= \begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & (.7)^k\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix} = \begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}2\\ (.7)^k\end{bmatrix}\\
&= \begin{bmatrix}4-(.7)^k\\ 2+(.7)^k\end{bmatrix}
\end{aligned}$$

and thus extracting, the terms, after k years the east island will have $4-(.7)^k$ million members, and the west will have a population of $2+(.7)^k$. One can see for any $\varepsilon > 0$ choosing $N = \ln(\varepsilon)/\ln(.7)$ means for all $n \geq N$ we have $(.7)^k < \ln(\varepsilon)$ and thus $|4-(.7)^k-4| < \varepsilon$. A similar argument can be done for w_k, and we can see the population stabilises over time to the populations $e_k = 4, w_k = 2$.

Exercise 3.3 Let

$$A = \begin{bmatrix}5 & 2\\ 2 & 8\end{bmatrix}$$

1. Find an orthogonal matrix P that diagonalises A.
2. Show that A is positive definite. Find a matrix S such that $S^2 = A$.
3. Find a matrix M such that $A = M^\top M$.

■

To find an orthogonal matrix that diagonalises A we find *any* matrix that diagonalises A, and because A is symmetric, it should be orthogonal. The characteristic equation for A is

$$\det(A - I\lambda) = \det\left(\begin{bmatrix}5-\lambda & 2\\ 2 & 8-\lambda\end{bmatrix}\right)(5-\lambda)(8-\lambda)-4$$

which when set to 0 gives $\lambda^2 - 13\lambda + 36 = (\lambda-9)(\lambda-4) = 0$ and thus $\lambda_1 = 9$, $\lambda_2 = 4$. The eigenvalues for λ_1, λ_2 happen when their characteristics things equal 0, so

$$(A-\lambda_1 I)v = \begin{bmatrix}5-9 & 2\\ 2 & 8-9\end{bmatrix}\begin{bmatrix}x_1\\ y_1\end{bmatrix} = \begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ y_1\end{bmatrix}$$

$$(A-\lambda_2 I)v = \begin{bmatrix}5-4 & 2\\ 2 & 8-4\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix}$$

luckily it can be easily seen that $x_1, y_1 = 1, 2$, and $x_2, y_2 = 2, -1$. We need to normalise these vectors. Notice $\|[x_1; y_1]\| = \sqrt{1^2+2^2} = \sqrt{5}$, and $\|[x_2; y_2]\| = \sqrt{5}$. So setting the columns of P as the normalised vectors we just calculated, and letting D be the diagonal matrix with entries of the eigenvalues gives

$$P = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix}9 & 0\\ 0 & 4\end{bmatrix}$$

Since our matrix is a square matrix that is symmetric and has all positive eigenvalues, we can say it is certainly positive definite. To find the "square root" of this matrix, we can "split" D into two parts and multiply on both sides so we get an even amount. We denote the "split" of D by Δ_i for $i \in \{1,2,3,4\}$. These are specifically:

$$\Delta_1 = \begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}, \Delta_2 = \begin{bmatrix}-3 & 0\\ 0 & -2\end{bmatrix}, \Delta_3 = \begin{bmatrix}-3 & 0\\ 0 & 2\end{bmatrix}, \Delta_4 = \begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}$$

So our possible candidates for S are

$$S_1 = \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$
$$S_2 = \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}-3 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$
$$S_3 = \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}-3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$
$$S_4 = \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$

I'm pretty sure we only need to find "a" matrix whose square is A, so we will calculate S_1 by

$$\frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}3 & 6\\ 4 & -2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}11 & 2\\ 2 & 14\end{bmatrix}$$

Indeed this is equal to A when squared. To solve (3), we will use $M = \Delta P^\top = \Delta P$, where $\Delta = \sqrt{D}$. This yields

$$M = \frac{1}{\sqrt{5}}\underbrace{\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}}_{\sqrt{D}}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}3 & 6\\ 4 & -2\end{bmatrix}$$

Exercise 3.4 Let

$$A = \begin{bmatrix}-5 & 0\\ 4 & 7\\ 3 & 1\end{bmatrix}$$

1. Calculate $||A||_F$.
2. Calculate $||A||_\infty$.
3. Calculate $||A||_1$.
4. Calculate $||A||_2$.
5. Show the inequality $||Ax||_2 \leq ||A||_2||x||_2$ holds for $x = \begin{bmatrix}3\\ 4\end{bmatrix}$.

The question (1) is just the Forbenius norm which can be calculated as

$$||A||_F = \sqrt{25+16+49+9+1} = \sqrt{100} = 10$$
$$||A||_\infty = \max\{5, 11, 4\} = 11$$
$$||A||_1 = \max\{12, 8\} = 12$$

For (4), we need to recall the definition of the 2 norm of a matrix.

Definition 3.1.1 Let A be a $n \times m$ matrix. Then

$$||A||_2 := \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^\top A\}$$

We clearly need to find the eigenvalues of $A^\top A$ first, so our first step is to calculate the characteristic polynomial $\det(A^\top A - \lambda I)$

$$\det(A^\top A - \lambda I) = \det\left(\begin{bmatrix}-5 & 4 & 3\\ 0 & 7 & 1\end{bmatrix}\begin{bmatrix}-5 & 0\\ 4 & 7\\ 3 & 1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) = \begin{bmatrix}50-\lambda & 31\\ 31 & 50-\lambda\end{bmatrix}$$

which is $(50-\lambda)(50-\lambda)-(31)(31) = (50+31-\lambda)(50-31-\lambda) = (81-\lambda)(19-\lambda)$ which when set to zero gives $\lambda_1 = 81, \lambda_2 = 19$. Our answer is therefore $\|A\|_2 = \sqrt{81} = 9$. For (5) first note $\|x\|_2 = \sqrt{9+16} = \sqrt{25} = 5$. It makes sense to first calculate Ax:

$$\|Ax\|_2 = \left\| \begin{bmatrix} -5 & 0 \\ 4 & 7 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -15 \\ 40 \\ 13 \end{bmatrix} \right\| = \sqrt{225+1600+169} = \sqrt{1994}$$

From what we have gathered, the expression $\|Ax\|_2 \leq \|A\|_2 \|x\|_2$ simplifies by

$$\underbrace{\|Ax\|_2}_{\sqrt{1994}} \leq \underbrace{\|A\|_2}_{9} \underbrace{\|x\|_2}_{5}$$

and calculating $\sqrt{1994} \approx 44.6542271$, we can see this is marginally less than 45, but the inequality still holds.

Exercise 3.5 Let

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 30 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 9 \\ 31 \end{bmatrix}$$

and use the 1-norm in answering the following questions:

1. For $A = A_1$:
   i. Solve the system $Ax = b$ and the perturbed system $A\hat{x} = \hat{b}$.
   ii. Calculate the relative error $\frac{\|\delta x\|}{\|x\|}$ and the relative residual $\frac{\|\delta b\|}{\|b\|}$.
   iii. Find the condition number of A, and check that

$$\frac{\|\delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\delta b\|}{\|b\|}$$

2. Repeat part (a) with $A = A_2$.
3. Which matrix $A = A_1$ or A_2 is less susceptible to error when solving $Ax = b$, if the entries of b are subject to rounding or measurement error? Why?

■

Let's start with 1(i). From observation it can be seen that x will have to be equal to $x = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ (similar to the example in lecture), and $\hat{x} = \begin{bmatrix} 13 \\ -2 \end{bmatrix}$ which can be found by row reducing the bottom by 3 times the top. For part (ii), the relative error can be calculated as

$$\frac{\|\delta x\|}{\|x\|} = \frac{\|\delta x\|_1}{\|x\|_1} = \frac{\|x-\hat{x}\|_1}{|10+0|} = \frac{3+2}{10} = \frac{5}{10} = 50\%$$

And for $\delta b = b - \hat{b} = [1; -1]$, with $\|b\|_1 = 40$ Then

$$\frac{\|\delta b\|}{\|b\|} = \frac{|1|+|-1|}{40} = \frac{1}{20} = 5\%$$

To solve (iii), we need only utilise the formula $\text{cond}A_1 = \|A_1\|_1 \|A_1^{-1}\|_1$. First note

$$A_1^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

See that $\|A_1^{-1}\|_1 = \max 2+3/2, 1+1/2 = 3.5$. Also $\|A_1\| = \max 4,6 = 6$. Multiplying these we get $\text{cond}A = 21$. To verify the inequality we have to observe $50\% \leq 21 \cdot 5\%$.

Moving on to (2). We do the exact same as we have previously done, save the differing matrix A_2. For (i), x is certainly the same, as $x = [10;0]$. Trying to solve for $\hat{x}$ in the head is a pain, so we will use elimination. See

$$\begin{bmatrix} 1 & 2 & 9 \\ 3 & -4 & 31 \end{bmatrix} \Longrightarrow_{R_2-3R_1} \begin{bmatrix} 1 & 2 & 9 \\ 0 & -10 & 4 \end{bmatrix} \quad \Longrightarrow \quad \hat{x}_2 = -\frac{2}{5}, \hat{x}_1 = 9 + \frac{4}{5} = \frac{49}{5}$$

Now for (ii), we first see $\delta x = \begin{bmatrix} 10 \\ 0 \end{bmatrix} - [49/5 - 2/5] = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$ whose 1-norm is $3/5 = .6$. Thus we have

$$\frac{\|\delta x\|}{\|x\|} = \frac{\|\delta x\|_1}{\|x\|_1} = \frac{.6}{10} = .06$$

To calculate the relative residual, we first see the relative residual is the same as in the last case, which was 5%. To find the condition number of A, we first find A_2^{-1}

$$A_2^{-1} = \frac{1}{-4-6} \begin{bmatrix} -4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-10} \begin{bmatrix} -4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 \\ 3/10 & -1/10 \end{bmatrix}$$

whose norm is $\|A_2^{-1}\|_1 = \max\{7/10, 3/10\} = 7/10$. Since $\|A_2\|_1 = \max\{4,6\} = 6$, then the condition number of A_2 is $7/10 * 6 = 4.2$, and clearly the equation

$$50\% \leq 4.2 \times .5\%$$

holds. By definition: the condition numbers measures how sensitive the answer is to possible perturbations in the input data and to roundoff errors so we can say that since $\text{cond}A_1 = 21$ and $\text{cond}A_2 = 4.1$, then A_1 is more sensitive to changes of input data.

3.2 Part Two - Octave/Matlab Questions

Exercise 3.6 Let

$$A = \begin{bmatrix} 2 & 2 \\ -5 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 13 & -1 & 2 \\ 6 & 7 & 5 \\ 6 & 8 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 & 3 \\ -1 & 4 & 2 \\ 1 & -5 & -3 \end{bmatrix}$$

Use ten iterations of the QR algorithm to estimate the eigenvalues of each of A, B, and C. Compare your approximate eigenvalues with the exact eigenvalues, found by hand or using the command `eig`.

Submit your commands and Octave/Matlab output. Clearly state your estimates for the eigenvalues, and make some comment about how they compare with the exact values.

Some hints. The command `[Q, R] = qr(A)` returns the QR-factorization of A. You can either write an m-file containing a for loop to iterate the QR algorithm, or you can use the up arrow to repeat an earlier command without re-typing it. If you use this latter approach, you may find it convenient to put several commands on a single line, separated by either commas or semi-colons, so that you can carry out an iteration of the algorithm in just one line. ■

We first show our output for **clock**.

```
>> clock

ans =

   1.0e+03 *

    2.0240    0.0100    0.0210         0    0.0210    0.0445

>>
```

Now we first present the actual eigenvalues of the matrices presented, using the command **eig**:

```
Command Window
>> A = [2 2 ; -5 9];
>> B = [13 -1 2 ; 6 7 5 ; 6 8 4];
>> C = [2 3 3 ; -1 4 2 ; 1 -5 -3];
>> eig(A), eig(B), eig(C)

ans =

     4
     7


ans =

   15.0000
   10.0000
   -1.0000


ans =

    2.0000
    2.0000
   -1.0000

>> 
```

To use the QR algorithm, we use the following code function, named **QRE**, for QR eigenvalues.

```
function A = QRE(M)

for i = 1:10

     [Q,R] = qr(M);    % Construct the QR decomposition
     M = R*Q;          % Set the new M as RQ, and repeat

end
A = M;                % The answer A is the final M
```

Using this command on our matrices gives

```
Command Window

>> QRE(A), QRE(B), QRE(C)

ans =

    7.0135    6.9942
   -0.0058    3.9865


ans =

   15.0712   -7.7315   -2.0936
    0.0467    9.9288   -3.0194
    0.0000    0.0000   -1.0000


ans =

    2.1961    1.3754   -6.3394
   -0.0277    1.8130    5.1761
    0.0000   -0.0049   -1.0091
```

The diagonals of these outputs are remarkably close to the eigenvalues of the respective matrices, with the maximum error "distance" being $\approx .2$.

Exercise 3.7 Let

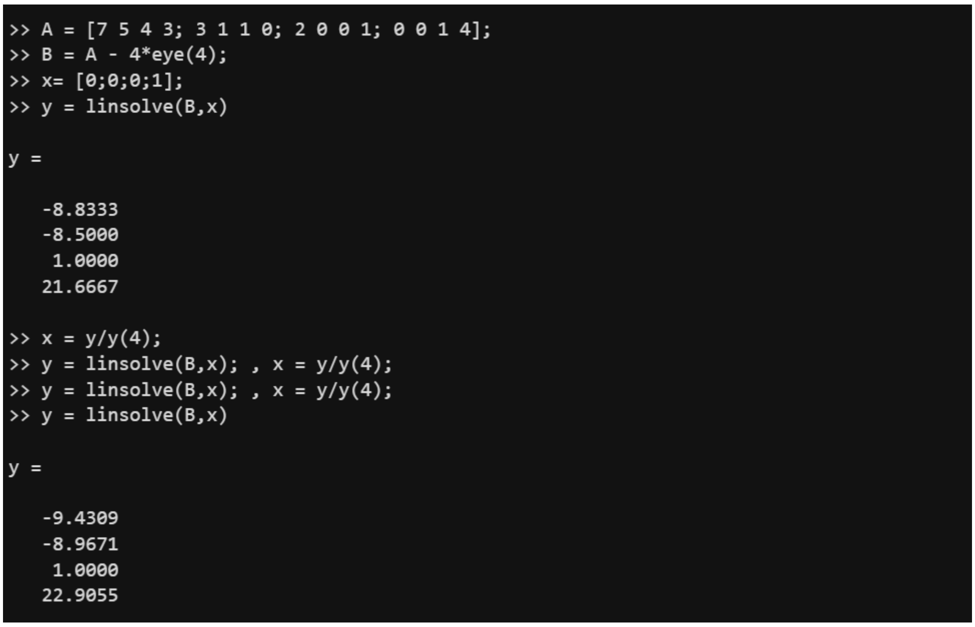
$$A = \begin{bmatrix} 7 & 5 & 4 & 3 \\ 3 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Given that 4 is an approximate eigenvalue of A with approximate eigenvector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

use Octave/Matlab to perform four steps of the shifted inverse power method to find improved approximations to the eigenvalue and eigenvector. ■

Here is the result that I get from Matlab.



```
>> A = [7 5 4 3; 3 1 1 0; 2 0 0 1; 0 0 1 4];
>> B = A - 4*eye(4);
>> x= [0;0;0;1];
>> y = linsolve(B,x)

y =

   -8.8333
   -8.5000
    1.0000
   21.6667

>> x = y/y(4);
>> y = linsolve(B,x); , x = y/y(4);
>> y = linsolve(B,x); , x = y/y(4);
>> y = linsolve(B,x)

y =

   -9.4309
   -8.9671
    1.0000
   22.9055
```

Let's walk through this:

```
>> A = [7 5 4 3; 3 1 1 0; 2 0 0 1; 0 0 1 4];  % Set A as the matrix
>> B = A - 4*eye(4);                           % Set B with a shift of 4
>> x= [0;0;0;1];            % Set x
>> y = linsolve(B,x)        % Solve for y

y =

   -8.8333
   -8.5000
    1.0000
   21.6667                  % Notice the last entry is the biggest

>> x = y/y(4);                                   % Set the new x
>> y = linsolve(B,x); , x = y/y(4);              % Repeat (2nd iterative)
>> y = linsolve(B,x); , x = y/y(4);              % Repeat (3rd iterative)
>> y = linsolve(B,x)                             % Repeat (last iterative)

y =

   -9.4309
   -8.9671
    1.0000
   22.9055

>> lambda = 4 + 1/eye(4) % Take back shift with reciprocal of largest entry
```

This gives $\lambda = 4.0437$. As shown:

```
>> lambda = 4 + 1/y(4)

lambda =

    4.0437

>> eig(A)

ans =

    9.7709
   -1.4233
   -0.3912
    4.0437
```

Nice!