I Differential Equations 1

1. Linear Differential Equations

> In order to solve a differential equation you look at it till a solution occurs to you.
>
> *George Pólya*

1.1 Structure of these notes

The following are some walkthrough problems for linear differential equations. Each exercise is put in an **Exercise** environment, with the answer explained after. If the question has multiple parts to it, the *final* solution to each part will be $\boxed{\text{boxed like this}}$.

1.2 Exercises

Exercise 1.1 Verify the equation

$$-2x^2y + y^2 = 1$$

is an implicit solution of

$$2xy + (x^2 - y)\frac{dy}{dx} = 0$$

■

Implicitly differentiating our solution can be carried out like so:

$$\begin{aligned} -2x^2y + y^2 &= 1 \\ -4xy - 2x^2y' + 2y \cdot y' &= 0 \\ -2xy - x^2y' + y \cdot y' &= 0 \\ 2xy + x^2y' - y \cdot y' &= 0 \\ 2xy + (x^2 - y)\frac{dy}{dx} &= 0 \end{aligned}$$

confirming the supposition.

Exercise 1.2 Write a differential equation expressing the following: On the graph $y = f(x)$, where the slope of the tangent at a point $P(x,y)$ is equal to the distance from $P(x,y)$ to the origin... ■

The distance from a point $(x,y) = (x, f(x))$ to $(0,0)$ is given by $\sqrt{x^2 + f(x)^2}$ which is equal to $\sqrt{x^2+y^2}$. The slope at (x,y) is given by $f'(x)$. So our differential equation would look like:

$$\frac{dy}{dx} = \sqrt{x^2+y^2}$$

Exercise 1.3 Find all values of m such that $y = x^m$ is a solution of the ODE:

$$xy'' + 7y' = 0$$

■

At first glance we have $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$ so simply substitute:

$$\begin{aligned}
xy'' + 7y' &= 0 \\
m(m-1)x^{m-1} + 7mx^{m-1} &= 0 \\
m^2x^{m-1} - mx^{m-1} + 7mx^{m-1} &= 0 \\
m^2 - m + 7m &= 0 \\
m^2 + 6m &= 0 \\
m(m+6) &= 0
\end{aligned}$$

therefore we must have $m \in \{-6, 0\}$.

Exercise 1.4 A model for the populations of two interacting species is:

$$\frac{dx}{dt} = kx(a-x)$$

$$\frac{dy}{dt} = mxy$$

for constants $k, a, m \in \mathbb{R}$. Solve the first equation and substitute this into the second one to get a single equation that y satisfies. ■

We solve the first one by separation of variables:

$$\begin{aligned}
\frac{1}{x(a-x)}\,dx &= k\,dt \\
\frac{1}{ax} + \frac{1}{a^2-ax}\,dx &= k\,dt \\
\frac{1}{a}\ln(x) - \frac{1}{a}\ln(a-x) &= kt + C_1 \\
\ln\left(\frac{x}{a-x}\right) &= akt + C_2 \\
\frac{x}{a-x} &= Ce^{akt} \\
x &= \frac{aCe^{akt}}{1+Ce^{akt}}
\end{aligned}$$

now we substitute this into the second equation:

$$\frac{dy}{dt} = m\frac{aCe^{akt}}{1+Ce^{akt}}y$$

Exercise 1.5 Solve the linear equations

$$(x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy$$

and

$$y' = 2y + x^2 + 5$$

■

Theorem 1.2.1 Given an ODE of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we find the **integrating factor** by $\mu(x) = e^{\int P(x)\,dx}$. Then the general solution $y(x)$ for this ODE can be solved for in the equation

$$\mu(x)y = \int \mu(x)Q(x)\,dx$$

For the first one, we first rearrange the ODE:

$$\frac{dy}{dx} + \frac{4}{(x+2)}y = \frac{5}{(x+2)^2}$$

so we find the integrating factor:

$$\mu(x) = \exp\left(\int \frac{4}{x+2}\,dx\right) = e^{4\ln(x+2)} = (x+2)^4$$

This clever trick:

$$\begin{aligned}
(x+2)^4\frac{dy}{dx} + 4(x+2)^3y &= 5(x+2)^2 \\
\frac{dy}{dx}\left[(x+2)^4 \cdot y\right] &= 5(x+2)^2 \\
\int \frac{dy}{dx}\left[(x+2)^4 \cdot y\right]\,dx &= \int 5(x+2)^2\,dx \\
(x+2)^4 y &= \frac{5}{3}(x+2)^3 + C \\
& \boxed{y = \frac{5}{3(x+2)} + \frac{C}{(x+2)^4}}
\end{aligned}$$

Turning our attention to the second ODE, we have put this in normal form:

$$\frac{dy}{dx} - 2y = x^2 + 5$$

with integrating factor $\mu(x) = e^{-2x}$, we multiply this by the equation, and use our trick:

$$e^{-2x}\frac{dy}{dx} - 2e^{-2x}y = (x^2+5)e^{-2x}$$
$$\frac{dy}{dx}\left[e^{-2x}\cdot y\right] = (x^2+5)e^{-2x}$$
$$\int \frac{dy}{dx}\left[e^{-2x}\cdot y\right]\,dx = \int (x^2+5)e^{-2x}\,dx$$
$$e^{-2x}y = -e^{-2x}\left(\frac{1}{2}x^2 + \frac{1}{2}x + \frac{11}{4}\right) + C$$
$$\boxed{y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4} + Ce^{2x}}$$

Exercise 1.6 Solve the initial value problem:

$$y'' - 2y' + y = 0 \qquad \text{and} \qquad y(0) = 5, y'(0) = 6$$

and the boundary value problem:

$$y'' - 2y' + 5y = 0 \qquad \text{and} \qquad y(0) = 1, y(3\pi/4) = 1$$

■

Fir the IVP we first will examine what a solution of $y = e^{mx}$ would look like to get:

$$m^2e^{mx} - 2me^{mx} + e^{mx} = 0$$
$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$m = 1$$

note that this $m = 1$ is repeated so our general solution will look like:

$$y = Ae^x + Bxe^x$$

Substituting the initial condition $y(0) = 5$ yields $A = 5$. See that

$$y' = (A+B)e^x + Bxe^x$$

so substituting $y'(0) = 6$ yields $A + B = 6$, and because we know $A = 5$, we now know that $B = 1$. This implies our solution to the IVP is given by

$$\boxed{y(t) = 5e^x + xe^x}$$

we now turn our attention to the BVP. The characteristic equation is given by

$$m^2 - 2m + 5 = 0$$
$$m^2 - 2m = -5$$
$$m^2 - 2m + 1 = -4$$
$$m - 1 = \pm 2i$$
$$m = 1 \pm 2i$$

thus our general solution is given by

$$y(t) = e^x (A\cos(2x) + B\sin(2x))$$

so substituting $y(0) = 1$ we will get $A = 1$. Substituting the condition $y(3\pi/4) = 1$ gives us

$$\begin{aligned} y(x) &= e^x (\cos(2x) + B\sin(2x)) \\ 1 &= e^{\frac{3\pi}{4}} \left(\cos\left(\frac{3\pi}{2}\right) + B\sin\left(\frac{3\pi}{2}\right)\right) \\ 1 &= -Be^{\frac{3\pi}{4}} \\ B &= -e^{-\frac{3\pi}{4}} \end{aligned}$$

so our solution to the BVP is:

$$\boxed{y(x) = e^x \left(\cos(2x) - \frac{1}{e^{3\pi/4}} \sin(2x)\right)}$$

Exercise 1.7 Solve the initial value problem

$$y'' + 4y' + 5y = 5e^{-4x} \qquad \text{and} \qquad y(0) = 3, y'(0) = -1$$

■

First find the complementary part:

$$\begin{aligned} m^2 + 4m + 5 &= 0 \\ m^2 + 4m &= -5 \\ (m+2)^2 &= -1 \\ m &= -2 \pm i \end{aligned}$$

which means our general (complementary) solution is

$$y_c = e^{-2x}(A\cos(x) + B\sin(x))$$

to find y_p we will want to try $y = Ce^{-4x}$. This would give us

$$\begin{aligned} 16Ce^{-4x} - 16Ce^{-4x} + 5Ce^{-4x} &= 5e^{-4x} \\ C &= 1 \end{aligned}$$

so in total our general solution to the differential equation is given by

$$y = y_p + y_c = e^{-4x} + e^{-2x}(A\cos(x) + B\sin(x))$$

now all the remains is to substitute the initial conditions. If we substitute $y(0) = 3$ into our general equation above, we will find $A = 2$. We will plug this into our general solution, then differentiate:

$$\begin{aligned} y &= e^{-4x} + e^{-2x}(2\cos(x) + B\sin(x)) \\ y' &= -4e^{-4x} + \left[-2e^{-2x}(2\cos(x) + B\sin(x)) + e^{-2x}(-2\sin(x) + B\cos(x))\right] \end{aligned}$$

lastly, substituting the initial condition $y'(0) = -1$ gives us

$$\begin{aligned} -1 &= -4 + [-2(2) + (B)] \\ -1 &= -4 + [-4 + B] \\ -1 &= -8 + B \\ B &= 7 \end{aligned}$$

thus, the solution for the IVP is:

$$y = e^{-4x} + e^{-2x}(2\cos(x) + 7\sin(x))$$

1.3 MATLAB Assignment

```
1   % visualise direction field for dy/dx=f(x,y). f(x,y)=x^2-y^2 in this case.
2   % make a mesh with specified x and y ranges
3
4   [x, y] = meshgrid(linspace(-2,2,10));
5
6   % calculate x and y direction field
7   xd = ones(size(x));
8   yd = x.^2-y.^2;
9
10  % plot direction field
11  quiver(x,y,xd,yd)
12 xlabel('x'); ylabel('y')
```

Exercise 1.8 The above is some Matlab code to visualize the direction field for a differential equation.

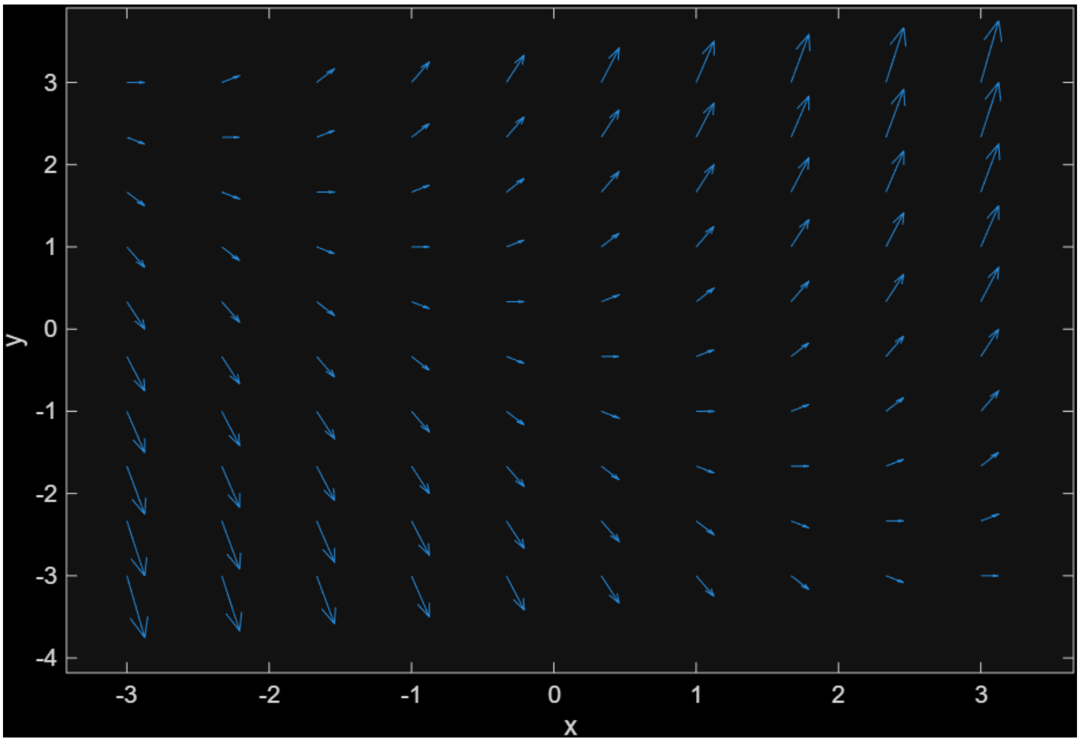
1. Modify the code to visualize the direction field for
$$\frac{dy}{dx} = x + y$$
2. Use a range $(x, y) \in [-3, 3]^2$. Save the code in an m-file.
3. Print the figure and by hand, sketch an approximate solution curve through $(-2, 0)$ and another one through $(0, 1)$.

■

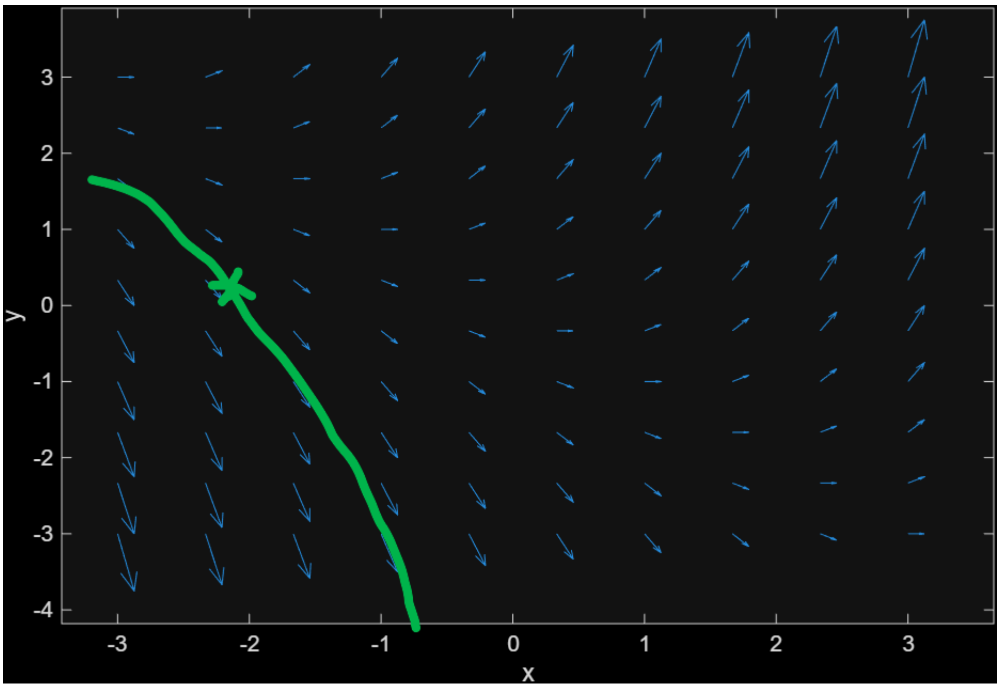
To modify the code, change line 8, to $yd = x + y$. If we want to use the 3×3 range our total program would now look like:

```
[x, y] = meshgrid(linspace(-3,3,10));

xd = ones(size(x));
yd = x + y;

quiver(x,y,xd,yd)
xlabel('x'); ylabel('y')
```

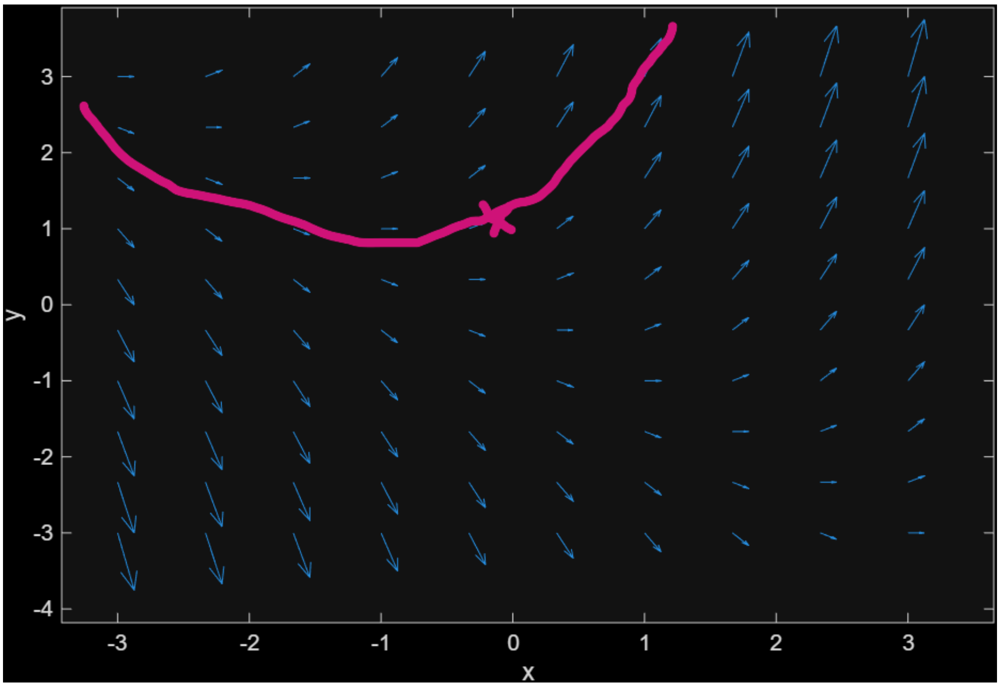
This code will output the following figure. Note that the x axis was made to accommodate the length of the lines, I'm not sure how I could get it to be the same size without changing a lot of the code.



Now the integral curve through the point $(-2,0)$ will roughly look like this:



and the integral curve through the point $(0,1)$ looks like:


3
2
1
0
-1
-2
-3
-4
y
-3
-2
-1
0
1
2
3
x

2. Additional techniques

> It's clearly a budget. It's got a lot of numbers in it.
>
> *George W. Bush*

2.1 Structure of these notes

We now take a look at some additional techniques for solving linear differential equations. Each exercise is put in an **Exercise** environment, with the answer explained after. If the question has multiple parts to it, the *final* solution to each part will be $\boxed{\text{boxed like this}}$.

2.2 Exercises

Exercise 2.1 Use variation of parameters to solve

$$2y'' + y' - y = e^x + 1$$

■

We are given a second order, non-homogeneous ODE. Let us solve the complementary y_c of

$$2y'' + y' - y = 0$$

The characteristic equation is $2m^2 + m - 1 = (2m-1)(m+1) = 0$ which gives roots $m = \frac{1}{2}$ and $m = -1$. This means our complementary solution is

$$y_c = Ae^{-x} + Be^{x/2}$$

which gives components $y_1 = e^{-x}$ and $y_2 = e^{x/2}$. We then compute the Wronskian as:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{x/2} \\ -e^{-x} & \frac{1}{2}e^{x/2} \end{vmatrix} = \left(\tfrac{1}{2}e^{x/2}e^{-x}\right) - \left(-e^{x/2}e^{-x}\right) = \frac{1}{2}e^{-x/2} + e^{-x/2} = \frac{3}{2}e^{-x/2}$$

We will now try to find the particular solution. This is of the form $y_p = u_1y_1 + u_2y_2$ which is then equal to $u_1e^{-x} + u_2e^{x/2}$. Now using the formulae for u_1' and u_2' we see

$$u_1' = -\frac{y_2 g}{aW} = -\frac{e^{x/2}(e^x+1)}{3e^{-x/2}} = -\frac{e^{2x} + e^x}{3}$$

$$u_2' = \frac{y_1 g}{aW} = \frac{e^{-x}(e^x+1)}{3e^{-x/2}} = \frac{e^{x/2} + e^{-x/2}}{3}$$

now to find u_1 and u_2 we integrate:

$$u_1 = \int u_1'\, dx = -\frac{1}{3}\int e^{2x} + e^x\, dx = -\frac{e^{2x}+2e^x}{6}$$
$$u_2 = \in u_2'\, dx = \frac{1}{3}\int e^{x/2} + e^{-x/2}\, dx = \frac{2(e^{x/2}-e^{-x/2})}{3}$$

which gives the final particular solution

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ &= -\frac{e^{2x}+2e^x}{6}e^{-x} + \frac{2(e^{x/2}-e^{-x/2})}{3}e^{x/2} \\ &= -\frac{e^x+2}{6} + \frac{2(e^x-1)}{3} \\ &= \frac{-e^x-2}{6} + \frac{4e^x-4}{6} \\ &= \frac{3e^x-6}{6} \\ &= \frac{1}{2}e^x - 1 \end{aligned}$$

so our solution to this ODE in total is given by

$$y_c + y_p = Ae^{-x} + Be^{x/2} + \frac{1}{2}e^x - 1$$

Exercise 2.2 Solve the boundary value problem

$$y'' - 2y' + 2y = e^{2x}$$

with $y(0) = 0$ and $y(\pi/2) = \pi$. ■

The characteristic of the associated homogenous equation is $m^2 - 2m + 2 = 0$. This is the same as $m^2 - 2m = -2$. And adding 1 to both sides gives $m^2 - 2m + 1 = -1$. This can be expressed as $(m-1)^2 = -1$ which implies $m - 1 = \pm i$. So our roots are $m = 1 \pm i$. Our complementary solution is thus

$$y_c = e^x\,(A\cos(x) + B\sin(x))$$

We now turn our attention to the right side $g(x) = e^{2x}$. So our guess to the solution is Ce^{2x}. See then

$$\begin{aligned} y' &= 2Ce^{2x} \\ y'' &= 4Ce^{2x} \\ y'' - 2y' + 2y &= 4Ce^{2x} - 4Ce^{2x} + 2Ce^{2x} \\ &= 2Ce^{2x} \end{aligned}$$

and since $y'' - 2y' + 2y = e^{2x}$ we have $2Ce^{2x} = e^{2x}$. This would mean $C = 1/2$ and thus our particular solution is $\frac{1}{2}e^{2x}$ and so our total solution to the ODE is

$$y = y_c + y_p = e^x\,(A\cos(x) + B\sin(x)) + \frac{1}{2}e^{2x}$$

Our only remaining problem is solving for A and B given our boundary conditions. For $y(0) = 0$ we first substitute $x = 0$ to get

$$y(x) = e^0(A+0) + \frac{1}{2}e^0 = A + \frac{1}{2}$$

And since this is equal to zero we have $A+\frac{1}{2}=0$ and thus $A=-\frac{1}{2}$. As for $y(\pi/2)=\pi$ we substitute $x=\pi/2$. This gives

$$\begin{aligned} y(\pi/2) &= e^{\pi/2}\left(-\frac{1}{2}\cdot 0 + B\cdot 1\right)+\frac{1}{2}e^{\pi} \\ &= e^{\pi/2}B+\frac{1}{2}e^{\pi} \end{aligned}$$

and since this is equal to 0 we have this equal to π we calculate:

$$\begin{aligned} e^{\pi/2}B+\frac{1}{2}e^{\pi} &= \pi \\ e^{\pi/2}B &= \pi-\frac{1}{2}e^{\pi} \\ B &= \pi e^{-\pi/2}-\frac{1}{2}e^{\pi/2} \end{aligned}$$

Now that we have A and B we can substitue these into our general solution to get

$$y = e^{x}\left[-\frac{1}{2}\cos(x)+\left(\pi e^{-\pi/2}-\frac{1}{2}e^{\pi/2}\right)\sin(x)\right]+\frac{1}{2}e^{2x}$$

Exercise 2.3 Solve the two initial value problems

$$x^2y''-5xy'+8y=0 \qquad \text{with} \qquad y(2)=20, \quad y'(2)=0$$

and

$$x^2y''-3xy'+4y=0 \qquad \text{with} \qquad y(1)=3, \quad y(1)=5$$

■

For the first problem we will subsitute $y=x^m$. This gives us

$$\begin{aligned} y' &= mx^{m-1} \\ y'' &= m(m-1)x^{m-2} \\ x^2y''-5xy'+8y &= x^2m(m-1)(x^{m-2})-5x(mx^{m-1})+8x^m \\ &= m(m-1)x^m-5mx^m+8x^m \end{aligned}$$

Now remember we have this whole thing equal to 0 so we get

$$\begin{aligned} m(m-1)x^m-5mx^m+8x^m &= 0 \\ m(m-1)-5m+8 &= 0 \\ m^2-m-5m+8 &= 0 \\ m^2-6m+8 &= 0 \\ (m-4)(m-2) &= 0 \end{aligned}$$

and so our roots are $m=4$ and $m=2$. This means the only non trivial solutions for $y(x)=x^m$ are $y(x)=x^2$ and $y(x)=x^4$. Since we have two distinct roots our general solution looks like

$$y = Ax^2+Bx^4$$

Now recall $y(2) = 20$. Susbstituting $x = 2$ gives

$$\begin{aligned} 20 &= 4A + 16B \\ 5 &= A + 4B \\ A &= 5 - 4B \end{aligned}$$

Now for our second equation, we will find $y'(x)$ and substitute $x = 2$:

$$\begin{aligned} y'(x) &= 2Ax + 4Bx^3 \\ y'(2) &= 4A + 32B \\ 0 &= 4A + 32B \\ 0 &= 4(5-4B) + 32B \\ 0 &= 20 - 16B + 32B \\ -20 &= 16B \\ B &= -\frac{5}{4} \end{aligned}$$

and using our expression of $A = 5 - 4B$ we have $A = 5 - (-5) = 10$. In summary $A = 10$ and $B = -5/4$. So the solution to the first IVP is

$$\boxed{y = 10x^2 - \frac{5}{4}x^4}$$

We move onto the second IVP. As before we will substitute $y = x^m$ into $x^2y'' - 3xy' + 4y = 0$ to get

$$\begin{aligned} x^2m(m-1)(x^{m-2}) - 3x(mx^{m-1}) + 4x^m &= 0 \\ m(m-1)x^m - 3mx^m + 4x^m &= 0 \\ m(m-1) - 3m + 4 &= 0 \\ m^2 - m - 3m + 4 &= 0 \\ m^2 - 4m + 4 &= 0 \\ (m-2)^2 &= 0 \end{aligned}$$

This gives us repeated roots of $m = 2$. In this case the general solution becomes

$$y = Ax^2 + Bx^2\ln(x)$$

We have an initial condition $y(1) = 3$ so lets substitue $x = 1$ to get

$$\begin{aligned} A + B\ln(1) &= 3 \\ A &= 3 \end{aligned}$$

which is quite nice. Now we find the derivative, and substitute $A = 3$ and $x = 1$ to get

$$\begin{aligned} y'(x) &= 2Ax + 2Bx\ln(x) + Bx \\ y'(1) &= 6 + 2B\ln(1) + B \\ &= 6 + B \end{aligned}$$

Now since $y'(1) = 5$ we have $5 = 6 + B$ so $B = -1$. In total our particular solution for the second IVP is

$$\boxed{y = 3x^2 - x^2\ln(x)}$$

Exercise 2.4 A mass of 1kg is attached to a spring with spring constant $2N/m$. The mass is released $1m$ below equilibrium with a upward velocity of $2m/s$. The mass experiences a damping equal to 2 times its instantaneous velocity.

- Find the position of the mass at time t if the mass is driven by an external force $f(t) = \cos(2t) + 3\sin(2t)$.
- Use Matlab or Octave to plot the position as a function of time for $0 \le t \le 10$.
- What is the amplitude of oscillation after a long time?

Since $m = 1$, $\gamma = 2$ and $k = 2$, the equation for this damped harmonic oscilator is given by

$$x'' + 2x' + 2x = \cos(2t) + 3\sin(2t)$$

where $x(t)$ denotes the distance of the mass from equilibrium. In this case $x(0) = -1$, and $x'(0) = 2$. We now want to solve for $x(t)$. The characteristic equation is given by $m^2 + 2m + 2 = 0$. Solving this like so:

$$\begin{aligned} m^2 + 2m + 2 &= 0 \\ m^2 + 2m &= -2 \\ m^2 + 2m + 1 &= -1 \\ (m+1)^2 &= -1 \\ m + 1 &= \pm i \\ m &= -1 \pm i \end{aligned}$$

which gives our complementary solution

$$y_c = e^{-t}\left(\alpha\cos(t) + \beta\sin(t)\right)$$

for constants α, β. Observing the right side $g(t) = f(t)$ we will try a particular solution of the form $x(t) = A\cos(2t) + B\sin(2t)$. See that we have derivatives

$$\begin{aligned} x(t) &= A\cos(2t) + B\sin(2t) \\ x'(t) &= -2A\sin(2t) + 2B\cos(2t) \\ x''(t) &= -4A\cos(2t) - 4B\sin(2t) \end{aligned}$$

so now we will substitute this into the ODE:

$$\begin{aligned} \cos(2t) + 3\sin(2t) &= x'' + 2x' + 2x \\ &= -4A\cos(2t) - 4B\sin(2t) + 2\left(-2A\sin(2t) + 2B\cos(2t)\right) + 2\left(A\cos(2t) + B\sin(2t)\right) \\ &= -4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t) + 2A\cos(2t) + 2B\sin(2t) \\ &= (-4A + 4B + 2A)\cos(2t) + (-4B - 4A + 2B)\sin(2t) \\ &= (4B - 2A)\cos(2t) + (-2B - 4A)\sin(2t) \end{aligned}$$

and matching the cofficents of the left hand side and right, we see

$$4B - 2A = 1 \qquad \text{and} \qquad -2B - 4A = 3$$

See that the first implies $B = (1+2A)/4$. If we substitute this into the second equation we get:

$$\begin{aligned} -\frac{1+2A}{2} - 4A &= 3 \\ \frac{-1-2A}{2} - \frac{8A}{2} &= 3 \\ \frac{-1-10A}{2} &= 3 \\ -1-10A &= 6 \\ -10A &= 7 \\ A &= -\frac{7}{10} \end{aligned}$$

and if we plug this value of A back into our expression of B we see

$$\begin{aligned} B &= \frac{1+2A}{4} \\ &= \frac{1-\frac{7}{5}}{4} \\ &= \frac{-\frac{2}{5}}{4} \\ &= -\frac{2}{20} = -\frac{1}{10} \end{aligned}$$

So our particular solution is

$$y_p = -\frac{7}{10}\cos(2t) - \frac{1}{10}\sin(2t)$$

and our total general solution is

$$x(t) = x_c + x_p = e^{-t}\left(\alpha\cos(t) + \beta\sin(t)\right) - \frac{7}{10}\cos(2t) - \frac{1}{10}\sin(2t)$$

We are not finished. We still have to solve for α and β using the initial conditions. We are given $x(0) = -1$, so let us substitute $t = 0$ to get

$$x(0) = \alpha - \frac{7}{10} = -1$$

so one will get $\alpha = 7/10 - 10/10 = -3/10$. This means our general solution is:

$$x(t) = e^{-t}\left(-\frac{3}{10}\cos(t) + \beta\sin(t)\right) - \frac{7}{10}\cos(2t) - \frac{1}{10}\sin(2t)$$

We will need to differentiate this in order to use the second initial condition $x'(0) = 2$. The derivative of the complementary solution $e^{-t}\left(-\frac{3}{10}\cos(t) + \beta\sin(t)\right)$ can be found with chain rule. We then add this to the derivative of the particular solution, to get the derivative of the general solution. This is done like so:

$$\begin{aligned} x_c' &= -e^{-t}\left(\alpha\cos(t) + \beta\sin(t)\right) + e^{-t}(-\alpha\sin(t) + \beta\cos(t)) \\ &= -e^{-t}\left(-\tfrac{3}{10}\cos(t) + \beta\sin(t)\right) + e^{-t}\left(\tfrac{3}{10}\sin(t) + \beta\cos(t)\right) \\ x_p' &= \frac{7}{5}\sin(2t) - \frac{1}{5}\cos(2t) \\ x' &= x_c' + x_p' \\ &= e^{-t}\left[\left(\frac{3}{10} + \beta\right)\cos(t) + \left(\frac{3}{10} - \beta\right)\sin(t)\right] + \frac{7}{5}\sin(2t) - \frac{1}{5}\cos(2t) \end{aligned}$$

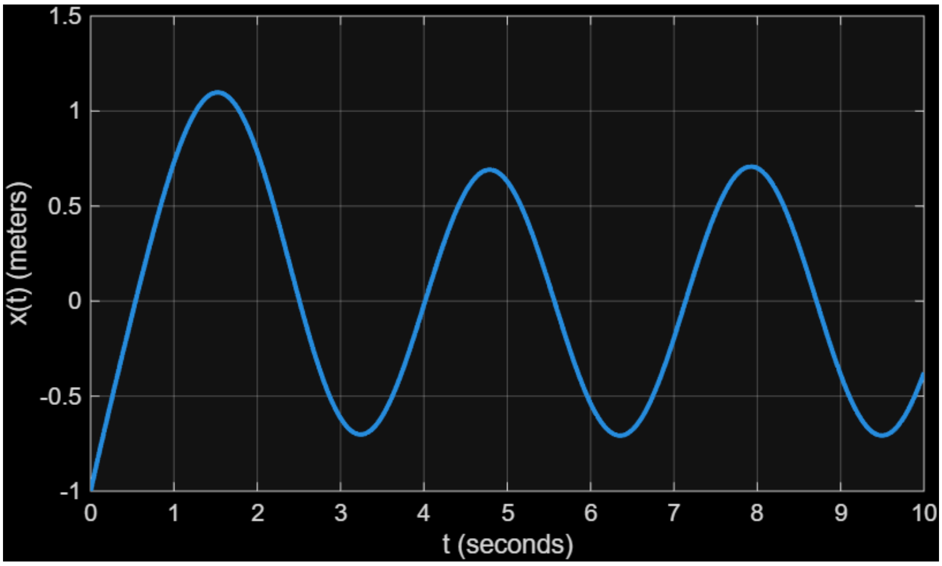
Now to use $x'(0) = 2$ we will substitute $t = 0$ into the equation above:

$$\begin{aligned} x'(0) &= \left(\frac{3}{10} + \beta\right) - \frac{1}{5} \\ &= \frac{3}{10} - \frac{2}{10} + \beta \\ &= \frac{1}{10} + \beta \end{aligned}$$

Since this is equal to 2 we have $2 = 1/10 + \beta$ which implies $20/10 - 1/10 = \beta$ and thus $\beta = 19/10$. This gives us the total general solution for this IVP as:

$$\boxed{x(t) = e^{-t}\left(-\frac{3}{10}\cos(t) + \frac{19}{10}\sin(t)\right) \ -\frac{7}{10}\cos(2t) - \frac{1}{10}\sin(2t)}$$

We leave it as an exercise to the reader to verify this is indeed a solution to the ODE.



The above is the plot made in MATLAB. It used the following code:

```
t = linspace(0,10,1000);
x = exp(-t).*(-0.3*cos(t) + 1.9*sin(t)) -0.7*cos(2*t) - 0.1*sin(2*t);
plot(t,x, 'LineWidth', 2)

xlabel('t (seconds)');
ylabel('x(t) (meters)');
```

A long time is mathematically defined as $t \to$. As t gets closer to infinity, the complementary part of the ODE goes to zero. We will be left with

$$-\frac{7}{10}\sin(2t) - \frac{1}{10}\cos(2t)$$

The amplitude of a function of form $A\sin(\omega t) + B\cos(\omega t)$ is simply $\sqrt{A^2 + B^2}$. So we have

$$\lim_{t\to\infty} \text{Amp}\left(x(t)\right) = \sqrt{\frac{49}{100} + \frac{1}{100}} = \frac{1}{\sqrt{2}}$$

Exercise 2.5 Find the general solution of the system

$$\frac{dx}{dt} = -4x + 5y$$
$$\frac{dy}{dt} = -5x + 4y$$

■

Let us fist solve for y in the first equation, and then differentiate like so:

$$\begin{aligned} y &= \frac{1}{5}\frac{dx}{dt} + \frac{4}{5}x \\ \frac{d}{dt}y &= \frac{d}{dt}\left(\frac{1}{5}\frac{dx}{dt} + \frac{4}{5}x\right) \\ \frac{dy}{dt} &= \frac{1}{5}\frac{d^2x}{dt^2} + \frac{4}{5}\frac{dx}{dt} \end{aligned}$$

Now we will substitute both expressions of y and dy/dt found above into the econd equation like so:

$$\begin{aligned} \frac{dy}{dt} &= -5x + 4y \\ \frac{1}{5}\frac{d^2x}{dt^2} + \frac{4}{5}\frac{dx}{dt} &= -5x + 4\left(\frac{1}{5}\frac{dx}{dt} + \frac{4}{5}x\right) \\ \frac{d^2x}{dt^2} + 4\frac{dx}{dt} &= -25x + 4\frac{dx}{dt} + 16x \\ \frac{d^2x}{dt^2} &= -9x \\ \frac{d^2x}{dt^2} + 9x &= 0 \end{aligned}$$

This is a second order homogenous equation of the form $x'' + 9x = 0$. Given $x(t) = e^{mt}$ its characteristic equation is $m^2e^{mt} + 9e^{mt} = 0$. This is the same as $m^2 + 9 = 0$ which has roots $m = \pm 3i$. This means our solution to $x'' + 9x = 0$ is given by

$$x(t) = A\cos(3t) + B\sin(3t)$$

and it's derivative is $x'(t) = -3A\sin(3t) + 3B\cos(3t)$. Becuase we have x and x', we can substitute this back into our equation for y:

$$\begin{aligned} y &= \frac{1}{5}\frac{dx}{dt} + \frac{4}{5}x \\ &= \frac{1}{5}(-3A\sin(3t) + 3B\cos(3t)) + \frac{4}{5}(A\cos(3t) + B\sin(3t)) \\ &= -\frac{3}{5}A\sin(3t) + \frac{3}{5}B\cos(3t) + \frac{4}{5}A\cos(3t) + \frac{4}{5}B\sin(3t) \\ &= \frac{3B+4A}{5}\cos(3t) + \frac{4B-3A}{5}\sin(3t) \end{aligned}$$

In summary, the solution to the system of ODEs given, is

$$\boxed{x(t) = A\cos(3t) + B\sin(3t)} \quad \text{and} \quad \boxed{y = \frac{3B+4A}{5}\cos(3t) + \frac{4B-3A}{5}\sin(3t)}$$

(R) Of course matrix methods could be applied to solve this, but it is more straightforward as presented.

2.3 MATLAB Assignment

Exercise 2.6 Consider the equation $X' = AX$ where

$$A = \begin{bmatrix} 11 & -6 & -2 \\ -20 & 18 & 4 \\ 120 & -90 & -23 \end{bmatrix}$$

1. Use Matlab to find the eigenvalues and eigenvectors of A. If possible, scale the eigenvectors so that their entries are all integers.
2. Use these results to write the general solution of $X' = AX$.
3. Determine the solution with initial condition

$$X(0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

   You may use Matlab to help with this calculation.
4. Use Matlab to plot the second component of this particular solution for $0 \leq t \leq 0.5$. (The solution is a vector;the second component is the second entry in the vector.)

■

The eigenvalue for A can be found by setting `A = [11, -6, -2; -20, 18, 4; 120, -90, -23];` and then displaying `eig(A)`. This will output a vector with entries $1, 2$, and 3. These are the eigenvalues of our matrix A. We will find the eigenvectors one by one. Suppose $\lambda_1 = 1$. Then we use the following code:

```
A = [11, -6, -2; -20, 18, 4; 120, -90, -23];

% One can swap this with 2 and 3
lambda = 1;

% Uses C as the "characteristic matrix"
C = A - lambda * eye(3);

% Displays the nullspace of C, in a rational basis
v1 = null(C, 'r');
disp(v1);
```

Doing this for all eigenvalues $1, 2$, and 3 output the eigenvectors:

$$v_1 = \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} .3333 \\ .16667 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/6 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} .75 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1 \\ 0 \end{bmatrix}$$

Where we have rewritten the output of MATLAB into fraction form. This makes it easy to see what the appropriate scalar is. In the first case we can myltiply by 5, in the second we can multiply by 6, and in the third by 4. THis gives us integer eigenvalues:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

For (2) the general solution of $X(t)$ is given by

$$X(t) = Ae^{t}\begin{bmatrix}1\\0\\5\end{bmatrix} + Be^{2t}\begin{bmatrix}2\\1\\6\end{bmatrix} + Ce^{3t}\begin{bmatrix}3\\4\\0\end{bmatrix}$$

For (3) simply substitute $t = 0$ to get the particular solution

$$X(0) = A\begin{bmatrix}1\\0\\5\end{bmatrix} + B\begin{bmatrix}2\\1\\6\end{bmatrix} + C\begin{bmatrix}3\\4\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

This gives the following silmutaneous equations:

$$\begin{cases} A + 2B + 3C = 0 \\ B + 4C = 1 \\ 5A + 6B = 0 \end{cases}$$

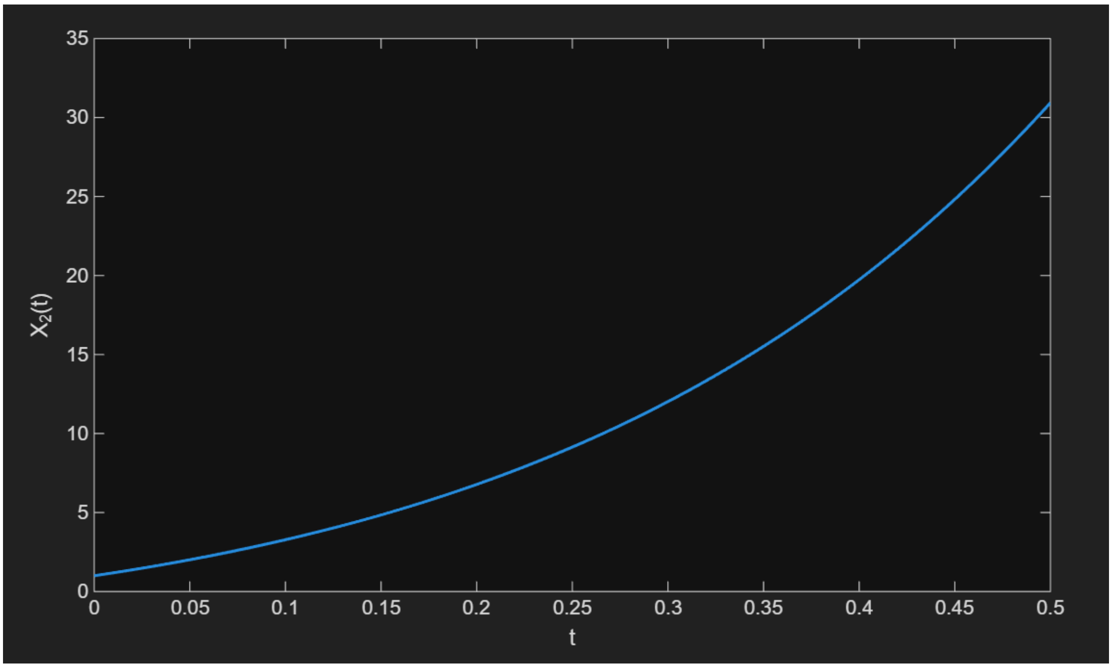
The second equation implies $B = 1 - 4C$. Substituting this in the first equation shows $A + 2 - 8C + 3C = A + 2 - 5C = 0$. This is the same as $A = 5C - 2$. Plugging this expression for A into the third equation gives $25C - 10 + 6 - 24C = 0$. This means $C = 4$. Now we know $A = 18$ and $B = -15$. In summary, the particular solution solving the IVP is given by

$$X(t) = 18e^{t}\begin{bmatrix}1\\0\\5\end{bmatrix} - 15e^{2t}\begin{bmatrix}2\\1\\6\end{bmatrix} + 4e^{3t}\begin{bmatrix}3\\4\\0\end{bmatrix}$$

The last question (4) asks us to plot the second component $x_2(t) = -15e^{2t} + 16e^{3t}$ for $0 \leq t \leq 0.5$ in MATLAB. To do this we use the following code:

```
t = linspace(0,0.5,1000);

x_2 = -15*exp(2*t) + 16*exp(3*t);


figure
plot(t,x_2,'LineWidth',1.5)
xlabel('t')
ylabel('X_2(t)')
```

Which gives the following output:



35
30
25
20
15
10
5
0
$X_2(t)$
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
t

3. MATLAB Assignment

Science is a differential equation.
Religion is a boundary condition.

Alan Turing

3.1 Structure of these notes

We now use MATLAB to explore some differential equations. Each exercise is put in an **Exercise** environment, with the answer explained after. If the question has multiple parts to it, the *final* solution to each part will be $\boxed{\text{boxed like this}}$.

This chapter is a little different to the previous ones. We have two main questions, each with sub-questions relating to the main question.

3.2 MATLAB Problems

3.2.1 Problem 1 : Yellow Bass Population

In 2010 a large number of yellow bass (fish) were introduced to a North American reservoir, but within a few years all perished. Scientists subsequently speculated that they were in competition for the same food source as zooplankton, that dominated the yellow bass. Based on this assumption they developed the model:

$$\begin{aligned} \frac{dy}{dt} &= y(3-y-z) \\ \frac{dz}{dt} &= z(4-2y-z) \end{aligned} \tag{3.1}$$

where $y(t)$ denotes the population of yellow bass in thousands, $z(t)$ denotes the population of zooplankton in millions, and t is time in years.

Exercise 3.1 Find the critical points of 3.1. ■

Recall the critical points are where $\frac{dy}{dt} = 0 = \frac{dz}{dt}$, which leads us to the obvious first critical point $(y,z) = (0,0)$. For the second, we set $y = 0$. This means we have $(4-z) = 0$ in which case $z = 4$ is a possible solution (we already accounted for $z = 0$). Now let's set $z = 0$, which leads us to $y(3-y) = 0$ by the first equation, and thus $y = 3$ is another non-trivial solution. Now suppose we set both equations equal to zero, then we have the two simultaneous equations $y+z=3$ and $4-2y-z=0$. The second can be rearranged to $2y+z=4$. Subtracting this from $y+z=3$ gives $-y=-1$ and thus $y=1$. This also implies $z=2$. To recap, our four critical points are

$$(y,z) \in \{(0,0) \quad , \quad (0,4) \quad , \quad (3,0) \quad ,(1,2)\}$$

> **Exercise 3.2** At each critical point, evaluate the trace and determinant of the Jacobian matrix of the system. Use these values to determine the stability of each critical point. ■

Let us calculate the derivative of the functions. Here we have

$$f(y,z) = 3y - y^2 - yz$$
$$g(y,z) = 4z - 2yz - z^2$$

with partial derivatives:

$$\frac{\partial f}{\partial y} = 3 - 2y - z, \quad \frac{\partial f}{\partial z} = -y,$$
$$\frac{\partial g}{\partial y} = -2z, \quad \frac{\partial g}{\partial z} = 4 - 2y - 2z$$

so the Jacobian of this system is given as such:

$$J = \begin{bmatrix} 3-2y-z & -y \\ -2z & 4-2y-2z \end{bmatrix}$$

Let's take the trivial point $(0,0)$. In this case we have the Jacobian equal to

$$J = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

We do not need the trace and determinant since one can see the eigenvalues are 3 and 4. Since the eigenvalues are both real and positive, we have an unstable node. Now look at the point $(0,4)$, the jacobian becomes:

$$J = \begin{bmatrix} -1 & 0 \\ -8 & -4 \end{bmatrix}$$

whose trace is -5 and its determinant is 4. A positive determinant means both eigenvalues are the same sign, and since we have a negative trace, we know both eigenvalues are negative. We therefore have a stable node at $(0,4)$. Now lets look at $(3,0)$, whose Jacobian is:

$$J = \begin{bmatrix} -3 & -3 \\ 0 & -2 \end{bmatrix}$$

whose trace is -5, and determinant is 6. So we have both eigenvalues being negative. This means the point $(3,0)$ is stable as well. Lastly we shall examine the point $(1,2)$. The Jacobian is

$$J = \begin{bmatrix} -1 & -1 \\ -4 & -2 \end{bmatrix}$$

which has trace -3 and determinant -2. Since the determinant is negative, the eigenvalues have opposite signs, making the point $(1,2)$ a saddle point. In summary $(0,0)$ is unstable. Both $(0,4)$ and $(3,0)$ are stable. Lastly, $(1,2)$ is a saddle point. These can be shown in the next exercise.

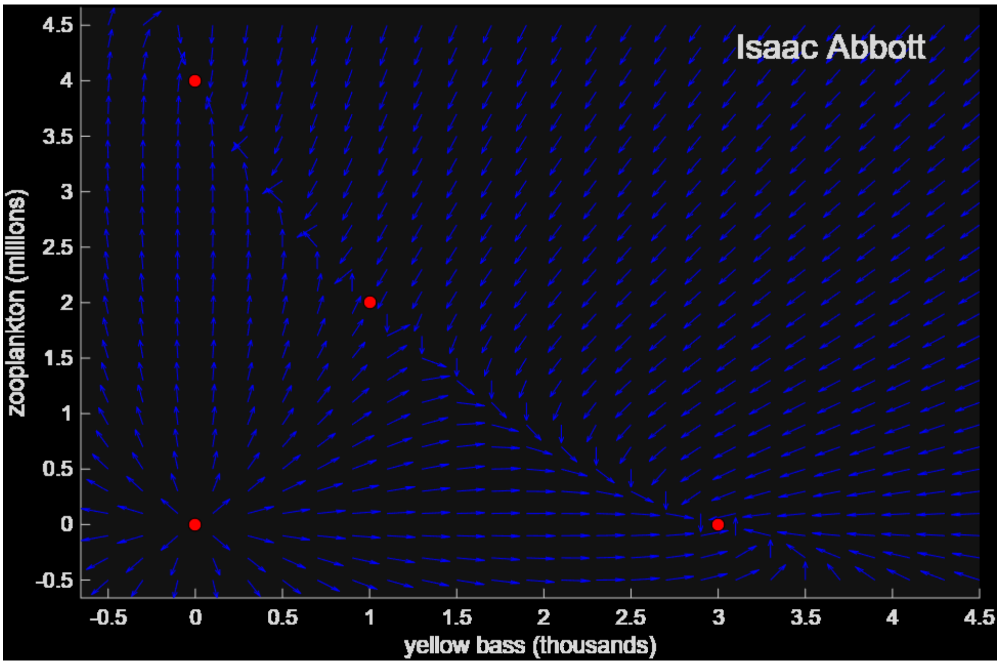
Exercise 3.3 Use matlab or octave to draw a vector field of 3.1 for a suitable range of values $y \geq 0$ and $z \geq 0$.

- Label the axes.
- Put a large dot at each critical point (or indicate the location of the critical points in some other way).
- Use the command `text` to add your name to the plot.
- Make sure to submit your code.

■

```
figure(1)
clf
hold on

% Plot from -.5 to 4.5 to better see what happens at the critical points:

[y,z] = meshgrid(-.5:0.2:4.5, -.5:0.2:4.5);

% System of ODEs:

dy = y .* (3 - y - z);
dz = z .* (4 - 2*y - z);

% To make each vector unit length:

L = sqrt(dy.^2 + dz.^2);
dy = dy ./ L;
dz = dz ./ L;

quiver(y, z, dy, dz, .6 , 'b')

% To plot critical points:
plot(0, 0, 'ko', 'MarkerSize', 6, 'MarkerFaceColor', 'r')
plot(0, 4, 'ko', 'MarkerSize', 6, 'MarkerFaceColor', 'r')
plot(3, 0, 'ko', 'MarkerSize', 6, 'MarkerFaceColor', 'r')
plot(1, 2, 'ko', 'MarkerSize', 6, 'MarkerFaceColor', 'r')

% Labels:

xlabel('yellow bass (thousands)')
ylabel('zooplankton (millions)')

text(3.1, 4.3, 'Isaac Abbott', 'FontSize', 16)
axis tight
```

Which gives the following output:

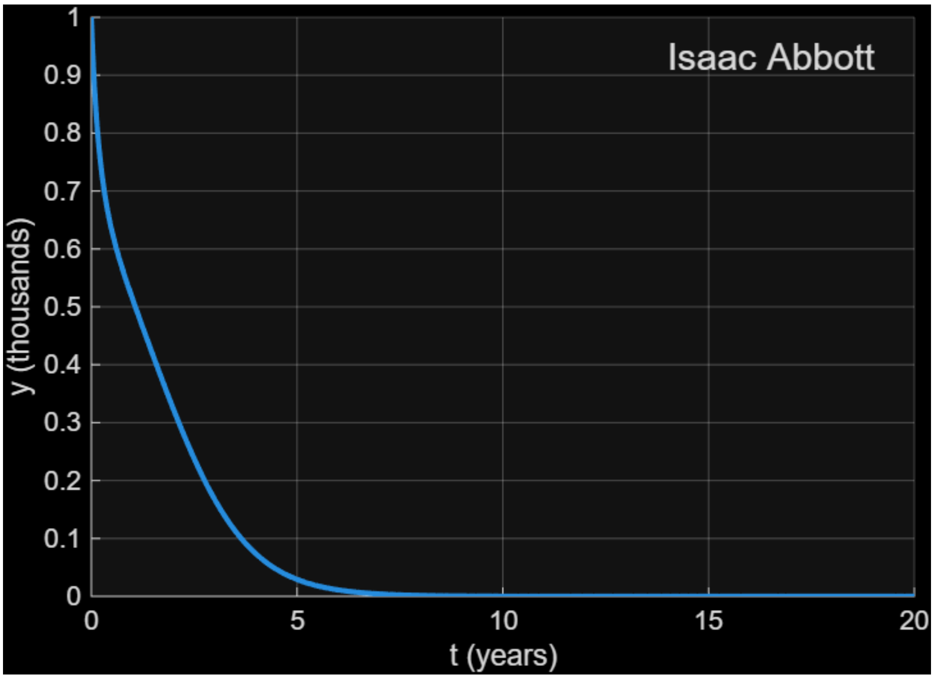


> **Exercise 3.4** Suppose one thousand yellow bass were introduced, and four million zooplankton were initially present. Use the command ODE45 to compute the solution to 3.1 with this initial condition. You should observe $y(t) \to 0$ as $t \to \infty$ (i.e. the yellow bass die out).
> - Plot the solution as a time series (with t on the horizontal axis and y on the vertical axis)
> - Make sure to submit your code
> - Based on this solution, how many yellow bass were remaining in the reservoir in 2016?

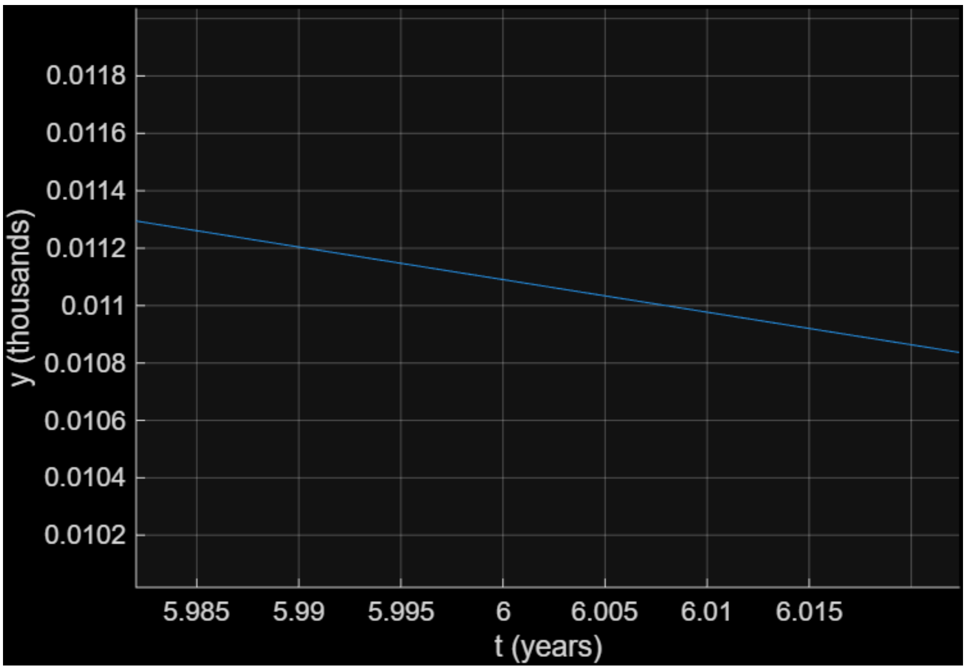
We simply use the following code:

```
figure(1)
clf
hold on


tMax = 20;
[T, X] = ode45(@f, [0, tMax], [1, 4]);
plot(T, X(:,1), 'LineWidth', 2)
xlabel('t (years)')
ylabel('y (thousands)')
text(14, 0.93, 'Isaac Abbott', "FontSize",14)

grid on
function dX = f(~, X)
    y = X(1);
    z = X(2);
    dy = y * (3 - y - z);
    dz = z * (4 - 2*y - z);
    dX = [dy; dz];
end
```

And we get this figure:



Now we are asked what happened in 2016, or at $t = 6$. After a lot of zooming into the graph one can see this:



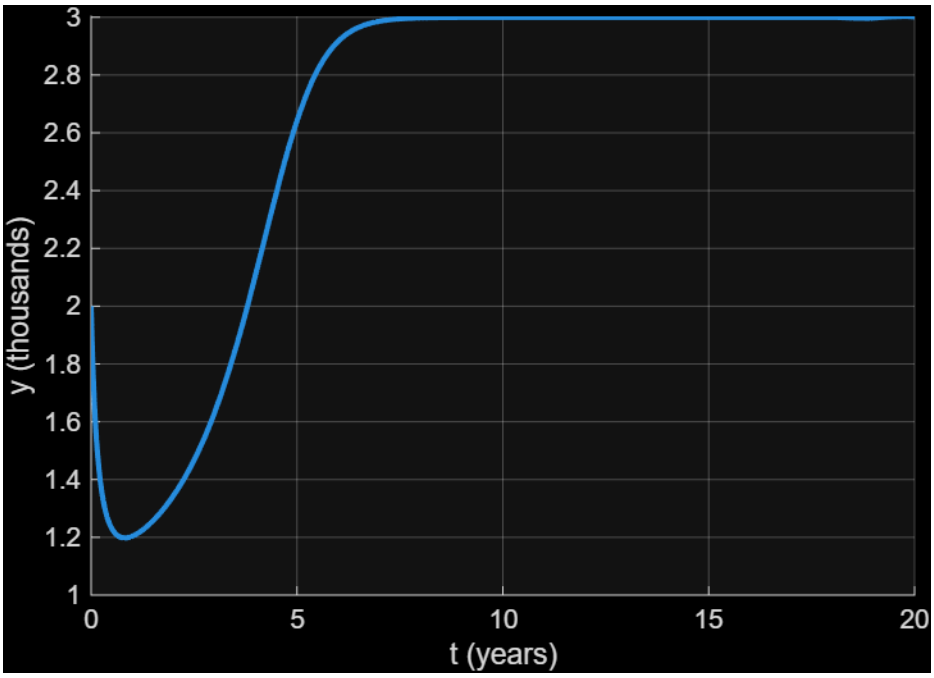
One can see it's a little over 0.011 thousand yellow fish remaining in 2016. Our once-thriving population of 1000 yellow bass has been decimated to a mere 11 after six years!

Exercise 3.5 Suppose instead two thousand yellow bass had been introduced (still with four million zooplankton initially present).

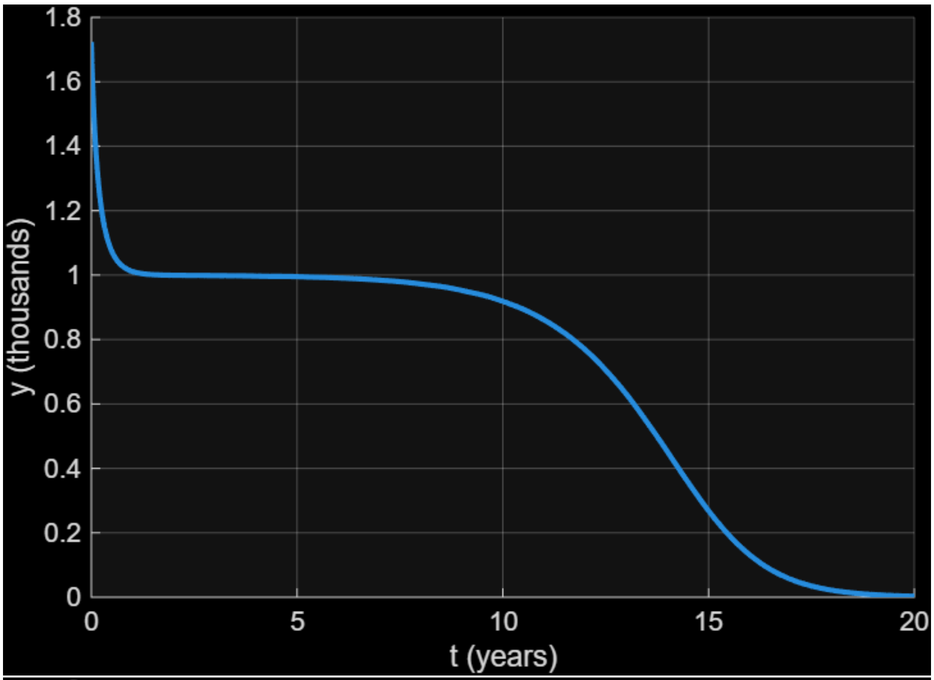
- Use ODE45 to plot the time series for this alternate initial condition.
- For this solution what is the value of $y(t)$ in the limit $t \to \infty$? (You should observe that the yellow bass do not die out.)
- By either a visual inspection of your vector field, or using ODE45 to explore the long-term behaviour of the system for various initial values, estimate the minimum number of yellow bass that need to be introduced for them to not die out.

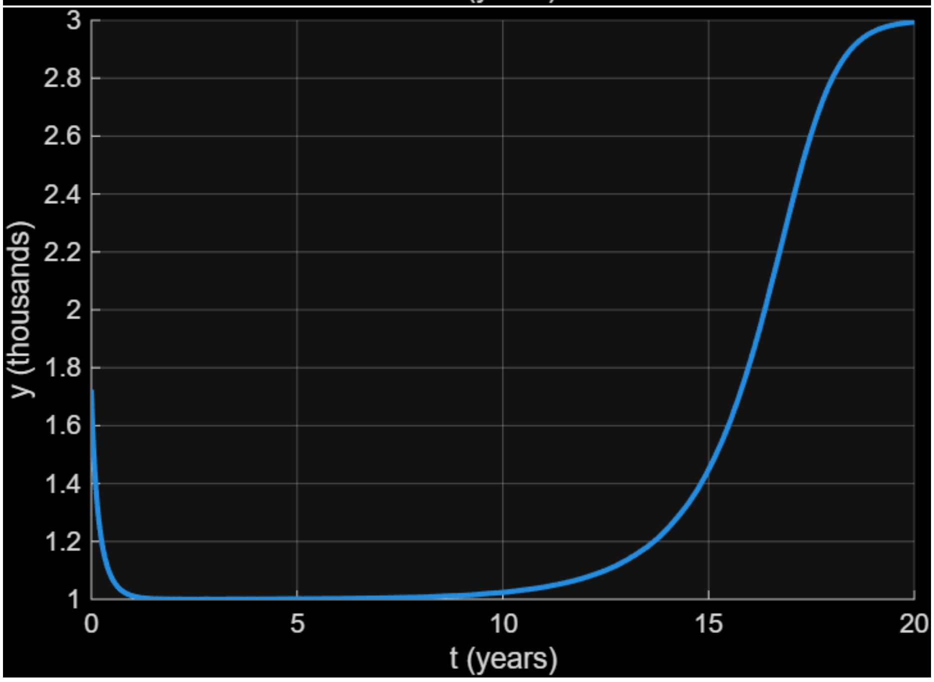
■

We use the same code from before but change `[T, X] = ode45(@f, [0, tMax], [1, 4]);` to `[T, X] = ode45(@f, [0, tMax], [2, 4]);` and we will get this plot:



Here one will notice $y(t) \to 3$ as $t \to \infty$. After fiddling around with the values, I have found 1.723 thousand yellow bass will lead to extinction, but with 1.724 thousand inital yellow bass,they will continue to live. Here are some plots:





The first one shows the fish will die if we only have 1,723 fish. If we have 1,724 fish though, the second plot says they will still live. So 1724 yellow bass is the minimum required to be introduced for them not to die out.

3.2.2 Problem 2 : Weka vs Possum

In a certain NZ forest the populations of weka (prey) and possums (predator) obey

$$\begin{aligned} \frac{dW}{dt} &= W\left(1 - \frac{W}{K} - \frac{3P}{W+2}\right), \\ \frac{dP}{dt} &= 3P\left(\frac{2W}{W+2} - 1\right), \end{aligned} \tag{3.2}$$

where $W(t)$ denotes the weka population (in thousands), $P(t)$ denotes the possum population (in thousands), t is time (in years), and the constant $K>0$ (in thousands) is the *carrying capacity* of the weka.

> **Exercise 3.6** With $K=4$ the system has a single critical point $(W^\star,P^\star)$ in the first quadrant $(W>0,P>0)$. Compute this critical point. ■

We set both equations to 0. Since we are only considering strictly positive values for W we have from the second equation:

$$\begin{aligned}\frac{2W}{W+2}&=1\\2W&=W+2\\2W-W&=2\\W&=2\end{aligned}$$

We now substitute this into the second equation becomes:

$$\begin{aligned}1-\frac{2}{K}-\frac{3P}{4}&=0\\\frac{8}{4K}+\frac{3PK}{4K}&=1\\\frac{8+3PK}{4K}&=1\\8+3PK&=4K\\3PK&=4K-8\\P&=\frac{4K-8}{3K}\end{aligned}$$

and since $K=4$ we have $P=2/3$. The critical point is thus $(W^\star,P^\star)=(2,\frac{2}{3})$.

> **Exercise 3.7** With $K=4$ evaluate the Jacobian matrix of 3.2 at $(W^\star,P^\star)$, and evaluate the eigenvalues of this matrix. Classify the critical point as either a stable node, a stable spiral, an unstable node, an unstable spiral, or a saddle. ■

To find the Jacobian, first set

$$\begin{aligned}f(W,P)&=W\left(1-\frac{W}{4}-\frac{3P}{W+2}\right)\\g(W,P)&=3P\left(\frac{2W}{W+2}-1\right)\end{aligned}$$

And calculate the partial derivatives of $f(W,P)$:

$$\begin{aligned}\frac{\partial f}{\partial W}&=W\frac{\partial}{\partial W}\left(1-\frac{W}{4}-\frac{3P}{W+2}\right)+\left(1-\frac{W}{4}-\frac{3P}{W+2}\right)\\&=W\left(-\frac{1}{4}-3P\frac{\partial}{\partial W}(W+2)^{-1}\right)+\left(1-\frac{W}{4}-\frac{3P}{W+2}\right)\\&=W\left(-\frac{1}{4}+\frac{3P}{(W+2)^2}\right)+\left(1-\frac{W}{4}-\frac{3P}{W+2}\right)\\&=-\frac{W}{2}+\frac{3PW}{(W+2)^2}-\frac{3P}{W+2}+1\\\frac{\partial f}{\partial P}&=-\frac{3W}{W+2}\end{aligned}$$

And likewise for $g(W,P)$:

$$\begin{aligned}
\frac{\partial g}{W} &= \frac{\partial}{\partial W}\left(6WP(W+2)^{-1}\right)\\
&= 6P(W+2)^{-1} - 6WP(W+2)^{-2}\\
&= \frac{6P}{W+2} - \frac{6WP}{(W+2)^2}\\
&= \frac{6P(W+2) - 6WP}{(W+2)^2}\\
&= \frac{6WP + 12P - 6WP}{(W+2)^2}\\
&= \frac{12P}{(W+2)^2}\\
\frac{\partial g}{\partial P} &= \frac{6W}{W+2} - \frac{3(W+2)}{(W+2)}\\
&= \frac{3W-6}{W+2}
\end{aligned}$$

The Jacobian is now given by

$$J(W,P) = \begin{bmatrix} -\frac{W}{2} + \frac{3PW}{(W+2)^2} - \frac{3P}{W+2} + 1 & -\frac{3W}{W+2} \\ \frac{12P}{(W+2)^2} & \frac{3W-6}{W+2} \end{bmatrix}$$

And we now tediously evaluate this at $(W,P) = (2, \frac{2}{3})$:

$$\begin{aligned}
\frac{\partial f}{\partial W}\Big|_{(2,\frac{2}{3})} &= -\frac{2}{2} + \frac{4}{(2+2)^2} - \frac{2}{2+2} + 1 = -1 + \frac{4}{16} - \frac{2}{4} + 1 = -\frac{1}{4}\\
\frac{\partial f}{\partial P}\Big|_{(2,\frac{2}{3})} &= -\frac{6}{4} = -\frac{3}{2}\\
\frac{\partial g}{\partial W}\Big|_{(2,\frac{2}{3})} &= \frac{8}{(4)^2} = \frac{1}{2}\\
\frac{\partial g}{\partial W}\Big|_{(2,\frac{2}{3})} &= \frac{6-6}{2+2} = 0
\end{aligned}$$

Which means our Jacobian at the critical point is

$$J(2, \tfrac{2}{3}) = \begin{bmatrix} -\frac{1}{4} & \frac{3}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

Now to find its eigenvalues we have characteristic equation found by

$$\begin{vmatrix} -\lambda - \frac{1}{4} & -\frac{3}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = -\lambda(-\lambda - \tfrac{1}{4}) + \frac{3}{4} = \lambda^2 + \frac{1}{4}\lambda + \frac{3}{4}$$

Using the quadratic formula we get

$$\begin{aligned}
\lambda &= \frac{-\frac{1}{4} \pm \sqrt{\frac{1}{16} - 3}}{2}\\
&= \frac{-\frac{1}{4} \pm \sqrt{-\frac{47}{16}}}{2}\\
&= \frac{-\frac{1}{4} \pm \frac{\sqrt{47}}{4}i}{2}\\
&= -\frac{1}{8} \pm \frac{\sqrt{47}}{8}i
\end{aligned}$$

Now since we have complex eigenvalues, with negative real part, this point is a stable spiral.

Exercise 3.8 Suppose at $t = 0$ there are three thousand weka and one thousand possums. Still with $K = 4$, use ODE45 to compute the solution for this initial condition.

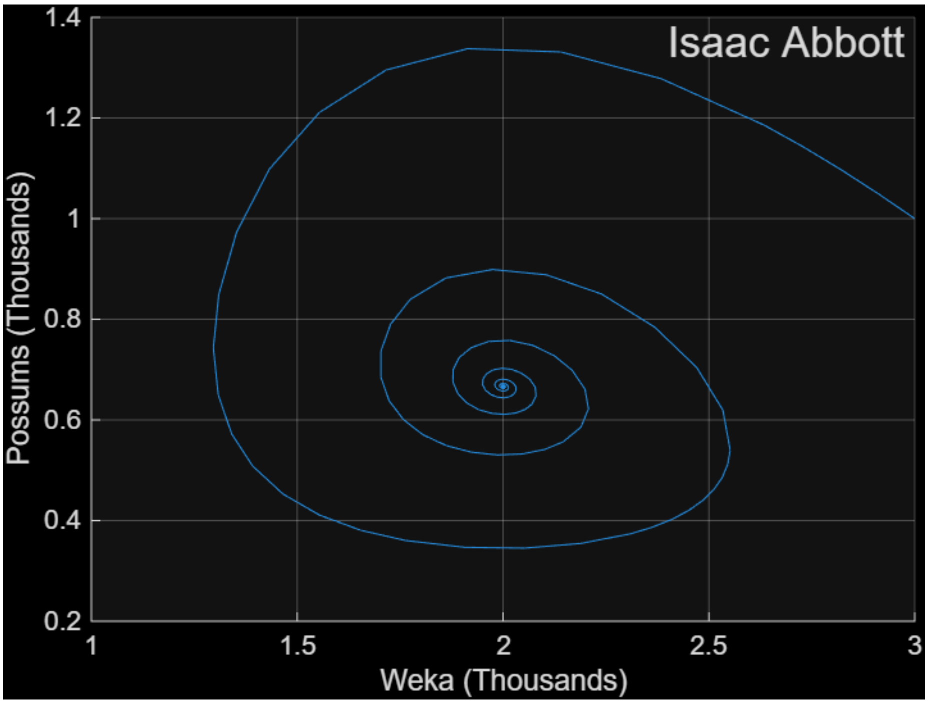
- Plot the solution in phase space (i.e. in the (W, P) plane).
- Label the axes.
- Use the command `text` to add your name to the plot.
- Make sure to submit your code.
- What is the behaviour of the weka and possum populations as $t \to \infty$? Explain how this is consistent with your answer to the previous answer.

■

In this case we set $K = 4$. Our intial conditions are $W = 3$ and $P = 1$, or the point $(3, 1)$. What we would expect is the solution to tend towards the point $(2, \frac{2}{3})$, in a spiral way because our previous answer said this point was a stable spiral. We use the following code:

```
% System of ODEs:

function dX = f(~, X)
    W = X(1);
    P = X(2);
    K = 4;
    dW = W * (1 - W/K - 3*P/(W+2));
    dP = 3*P * (2*W/(W+2) - 1);
    dX = [dW; dP];
end

figure(1)
clf
hold on

% Plot for reasonable tMax:

tMax = 100;
[T, X] = ode45(@f, [0, tMax], [3; 1]);
plot(X(:,1), X(:,2))


% Aesthetics
% Note that 'axis tight' isn't used because the plot would look ugly

xlabel('Weka (Thousands)')
ylabel('Possums (Thousands)')
grid on
text(2.4, 1.35, 'Isaac Abbott', 'FontSize', 16)
```

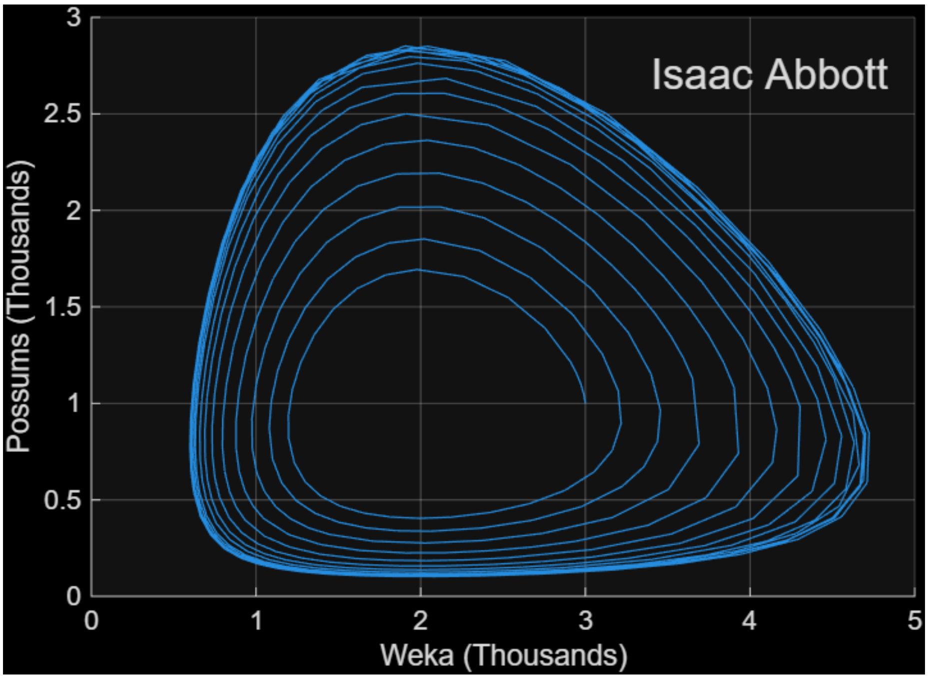
Which has the following output:



Which is *exactly* what we want to see. Nice!

Exercise 3.9 With instead $K = 7$ plot the solution in phase space with the initial condition given before. What is the behaviour of the weka and possum populations as $t \to \infty$? ■

Here all we do is modify the previous code from `K = 4` to `K = 7`. We also change the placement of my name from `text(2.4, 1.35...)` to `text(3.4, 2.7,...)`. The width of our plot is also changed to .6 to look nicer. Here we get:



Instead our solution is emanating from the initial condition outwards, and stabilizes into a spiral thing. This means when $K = 7$ the critical point becomes an instable spiral. Basically there is a never ending war between the Weka and Possums making the populations just cycle forever (war truly is terrible).

Exercise 3.10 For all $K > 2$ the system has a single critical point $(W_K^\star, P_K^\star)$ in the first quadrant.

1. Compute this critical point as a function of K.
2. Evaluate the trace, call it $\tau(K)$, and determinant of the Jacobian matrix at the critical point.
3. Solve $\tau(K) = 0$ for K.
4. Use your answers to (2) and (3) to predict the range of $K-$values for which typical solutions converge to $(W_K^\star, P_K^\star)$ for $t \to \infty$. Explain how this is consistent with your phase space plots for $K = 4$ and $K = 7$.

■

The second equation after being set to 0 gives us

$$\begin{aligned} 3P\left(\frac{2W}{W+2} - 1\right) &= 0 \\ \frac{2W}{W+2} &= 1 \\ 2W &= W + 2 \\ W &= 2 \end{aligned}$$

Which means the first component doesn't depend on K and is 2 regardless. The first equation of our system implies

$$\begin{aligned} 1 - \frac{2}{K} - \frac{3P}{4} &= 0 \\ \frac{3P}{4} &= 1 - \frac{2}{K} \\ P &= \frac{4}{3} - \frac{8}{3K} \\ P &= \frac{4K - 8}{3K} \end{aligned}$$

So our critical point for some $K > 2$ is given by

$$(W_K^\star, P_K^\star) = \left(2, \frac{4K-8}{3K}\right)$$

For (2) we first need to find the Jacobian, unfortunatley we have to do the following calculations:

$$\begin{aligned} \frac{\partial f}{\partial W} &= W\left(-\frac{1}{K} - 3P\frac{\partial}{\partial W}(W+2)^{-1}\right) + \left(1 - \frac{W}{K} - \frac{3P}{W+2}\right) \\ &= W\left(-\frac{1}{K} + \frac{3P}{(W+2)^2}\right) + \left(1 - \frac{W}{K} - \frac{3P}{W+2}\right) \\ \frac{\partial f}{\partial P} &= -\frac{3W}{W+2} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial g}{\partial W} &= \frac{\partial}{\partial W} 6PW(W+2)^{-1} \\
&= \frac{6P}{W+2} - \frac{6PW}{(W+2)^2} \\
&= \frac{6PW + 12P - 6PW}{(W+2)^2} \\
&= \frac{12P}{(W+2)^2} \\
\frac{\partial g}{\partial P} &= \frac{6W}{W+2} - 3 \\
&= \frac{3W-6}{W+2}
\end{aligned}$$

Now to evaulate the Jacobian at each point we substitute $W = 2$ and $P = \frac{4K-8}{3K}$. We get

$$\begin{aligned}
\left.\frac{\partial f}{\partial W}\right|_{(W_K^\star, P_K^\star)} &= -\frac{2}{K} + \frac{3P}{8} + 1 - \frac{2}{K} - \frac{3P}{4} \\
&= -\frac{4}{K} - \frac{3}{8}P + 1 \\
&= -\frac{4}{K} - \frac{K-2}{2K} + 1 \\
&= \frac{2K}{2K} - \frac{8}{2K} - \frac{K-2}{2K} \\
&= \frac{K-6}{2K} \\
\left.\frac{\partial f}{\partial P}\right|_{(W_K^\star, P_K^\star)} &= -\frac{6}{4} = -\frac{3}{2} \\
\left.\frac{\partial g}{\partial W}\right|_{(W_K^\star, P_K^\star)} &= \frac{12P}{16} = \frac{3}{4}P = \frac{12K-24}{12K} = \frac{K-2}{K} \\
\left.\frac{\partial g}{\partial P}\right|_{(W_K^\star, P_K^\star)} &= 0
\end{aligned}$$

So we have the Jacobian at the critical point given by

$$J(f,g)|_{(W_K^\star, P_K^\star)} = \begin{bmatrix} \frac{K-6}{2K} & -\frac{3}{2} \\ \frac{K-2}{K} & 0 \end{bmatrix}$$

From this we get it's trace and determinant is given by:

$$\begin{aligned}
\tau(K) &= \frac{K-6}{2K} \\
\det \begin{bmatrix} \frac{K-6}{2K} & -\frac{3}{2} \\ \frac{K-2}{K} & 0 \end{bmatrix} &= \frac{3(K-2)}{2K}
\end{aligned}$$

One can find $K = 6$ solves $\tau(K) = 0$ upon inspecting our expression for $\tau(K)$.
Now for (4) let's recap what we want. We want to know what values of K make the critical point stable. Remember for a stable point we need the real part of both eigenvalues of the Jacobian to be negative. To do this we need $\tau(K) < 0$. This is so that this forces the sum of the eigenvalues to be

negative. The only problem with this is that one of the eigenvalues could be positive and one could be negative. To fix this we also force $\det(J) > 0$. This means the product of the eigenvalues has to be positive, which can only happen when the real parts are the same sign. In summary to make the critical point stable we require both

$$\tau(K) < 0$$
$$\det(J) > 0$$

Which gives us

$$\frac{K-6}{2K} < 0$$
$$\frac{3(K-2)}{2K} > 0$$

The first implies $K < 6$, and the second implies $K > 2$, which is given. In summary, if we want the critical point to be stable, we require K be in the range $2 < K < 6$. This is consisten with our previous phase space plots for $K = 4$ and $K = 7$. At $K = 7$ we had an unstable point, this is because it was more than $K = 6$. Yet $K = 4$ was stable, because it was more than two but less than 6.