Combinatorics Notes (2025)

1. Counting Problems

> Combinatorics is the nanotechnology of mathematics.
>
> *Sara Billey*

1.1 Structure

These notes begin with counting problems involving student ID numbers under various restrictions, then moves to enumerating and comparing special seven-card hands in a 40-card deck. The remaining exercises cover counting set partitions, solving constrained integer equations via stars-and-bars, proving several binomial identities combinatorially, evaluating binomial sums using algebraic techniques, and concluding with a short power-series proof of the exponential law $e^x e^{-y} = e^{x-y}$.

1.2 Exercises

Exercise 1.1 Student ID numbers at a certain university consist of seven digits, taken from 0 to 9, and a single letter, taken from A to Z, for a total of eight characters. The letter can appear anywhere in the ID. Unless specified otherwise below, a digit can occur more than once in a student ID. *How many possible student ID numbers are there if:*

1. no digit may occur more than once?
2. exactly three of the digits must belong to $\{1,4,9\}$?
3. at most three of the digits may belong to $\{1,4,9\}$?
4. consecutive characters of the ID must be different

■

Lets do the first one. Suppose we ignore the letter, so we have a string of 7 *unique* digits, taken from 0-9 (10 choices). Then the possible number of IDs would be given by

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = \frac{10!}{3!}$$

Since we have to insert a letter from A-Z (26 options), in 8 possible places, our answer is

$$\frac{10!}{3!} \times 26 \times 8$$

For our second problem, if we ignore the letter, we have to choose exactly 3 numbers from $\{1,4,9\}$. This is with replacement since we can still choose duplicates, such as 1,1, and 4. This is $3\times3\times3 = 3^3$ choices. There are $\binom{7}{3}$ places to put these digits. Also, now we can't choose $1,4$ and 9 anymore, so we have 7 choices left for the rest. In summary we have

$$\overbrace{3\times3\times3}^{\text{digits from}\{1,4,9\}} \quad\times\quad \overbrace{\binom{7}{3}}^{\text{possible slots for these digits}} \quad\times\quad \overbrace{7\times7\times7\times7}^{\text{choices for remaining digits}}$$

Now recall we can choose from 8 slots where to put our letter, so our final answer is:

$$\binom{7}{3}\times 3^3\times 7^4\times 26\times 8$$

On our third problem, assume there is no letter. We break this into cases:

- **0 digits from** $\{1,4,9\}$. This is simply 7^7.
- **1 digit from** $\{1,4,9\}$. This is $7\times3\times7^6$.
- **2 digits from** $\{1,4,9\}$. This is also $\binom{7}{2}\times3^2\times7^5$.
- **3 digits from** $\{1,4,9\}$. Lastly, this is $\binom{7}{3}\times3^3\times7^4$.

And then we multiply by 26×8 for our letter. This gives our final (unsimplified answer) to be:

$$26\times8\times\left(\sum_{k=0}^{3}\binom{7}{k}\times3^k\times7^{7-k}\right)$$

Now this last one is a bit tricky. At first glance you might want to count how many IDs exist without a latter, with each consecutive character being different. However, we are actually allowed to have consecutive numbers, provided they are separated by the letter. It is more helpful to first look at the where the letter should go. Let's look at the cases:

- **Letter at the beginning.** Here we have 26 choices for the letter. Then 10 choices to choose the number. Because we can't have consecutive, we now have 9. This repeats. In this case we have $26\times10\times9^6$.
- **Letter in any of the middle.** In this case, we can have 10 choices for the number on the left, and 10 for the number on the right. This means we have $26\times10^2\times9^5$.
- **Letter at the end.** This is similar to our first case: $26\times10\times9^6$.

Recall that there are 6 total options for having a letter in the middle. So our final answer is:

$$2(26\times10\times9^6) \quad+\quad 6(26\times10^2\times9^5)$$

Exercise 1.2 Grumble is a card game played with a deck of 40 cards, consisting of the ace to 10 in each of the four suits hearts, diamonds, clubs, spades. A hand is an unordered selection without replacement of seven cards from the deck. A triple consists of three cards in a hand that are all the same rank, and that are not the same rank as any of the other cards in the hand; and likewise a pair consists of two cards in a hand that are the same rank, and that are not the same rank as any of the other cards in the hand.

1. How many different hands are there?
2. Mono is a hand that consists of seven cards that are all the same suit. How many mono hands are there?
3. Streak is a hand that consists of seven cards of consecutive ranks (the suits don't matter). How many streak hands are there?
4. Double trouble is a hand that consists of two triples, together with a card of some other

rank. How many double trouble hands are there?
5. Triceratops is a hand that consists of a triple and two pairs. How many triceratops hands are there?
6. Armadillo is a hand that consists of three pairs, together with a card of some other rank. How many armadillo hands are there?
7. Put the hands mono, streak, double trouble, triceratops and armadillo in order from most common to least common.

■

For (1), just choose any 7 cards of the 40. The answer is

$$\binom{40}{7}$$

as for (2), we divide our deck into 4 disjoint packs of 10, by suit. For each pack of suits you have $\binom{10}{7}$ ways to choose 7 cards. Multiplying this by 4 gives the answer:

$$4 \times \binom{10}{7}$$

now (3) requires a different approach. First see how many possible Streak hands can occur, regardless of suit. There are 4 possible such hands, starting from the hand $(\text{Ace}, \cdots 7)$ up to $(4, \ldots, 10)$. Now for each of these hands, we choose the possible suit wanted for each card, that is, 4^7. Multiplying gives our answer:

$$4^8$$

now question (4) has a similar procedure. At first, lets not consider suit. We have 10 options for the rank of the first triple, multiply this by $\binom{4}{3}$ for their possible suits, because their order doesn't matter. Then, the second triple has only 9 options of rank left, and we also multiply this by $\binom{4}{3}$ for the possible suits. The remaining card cannot be the same rank as the previous two, so that leaves 8 options, with 4 possible suits. We can multiply all this together, but remember that the order of the "first" triple, and "second" triple are irrelevant so we divide by $2! = 2$:

$$\frac{1}{2} \times [10 \times \binom{4}{3}] \times [9 \times \binom{4}{3}] \times [8 \times 4] = \binom{10}{2} \times \binom{4}{3}^2 \times 8 \times 4$$

For (5), we proceed in almost the exact same way. First choose 10 options of rank for the first triple, and multiply it by $\binom{4}{3}$ suits. Then for the two pairs, choose two ranks from the remaining 9: $\binom{9}{2}$, and multiply it by $\binom{4}{2}^2$ for the suits. In total:

$$10 \times \binom{4}{3} \times \binom{9}{2} \times \binom{4}{2}^2$$

Lastly, for Armadillo (6), we go through the same process to get:

$$\binom{10}{3} \times \binom{4}{2}^3 \times 7 \times 4$$

We now rank these hands by rarity, from most to least common. They are arranged as so:

Hand	Rarity	Chances of being delt this hand?
Armadillo	Basic hand	Odds of being unemployed in NZ with an undergrad degree.
Streak	Uncommon	Chances of flipping all heads eight times in a row.
Triceratops	Unusual	Probability of getting COVID-19 this week.
Double trouble	Rare	Likelihood of being born with green eyes.
Mono	Exotic	Probability of choosing a random word from *Animal Farm*.

Exercise 1.3 A class of 25 students is to be divided into four groups of size three, two groups of size four and a group of size five to work on a project. How many ways are there to do this if

1. the groups are labelled, such that groups 1 to 4 are to have size three, groups 5 and 6 are to have size four, and group 7 is to have size five?
2. the groups are unlabelled?

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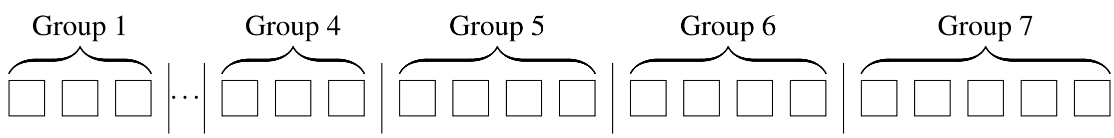
Suppose the groups are labelled, then we can visualize them like so:



Here, we have $25 \times 24 \times 23$ ways to choose what students go in group 1. But each group of 3 students are the same, so we divide by 3!. We now have 22 students left, so we have $22 \times 21 \times 20$ ways to choose what next students go into group 2. As before we have do divide by 3! because we don't care about the order. We keep doing this to get:

$$\frac{25 \times 24 \times 23}{3!} \cdot \frac{22 \times 21 \times 20}{3!} \cdots \frac{5 \times 4 \times 3 \times 2 \times 1}{5!} = \frac{25!}{(3!)^4 (4!)^2 5!}$$

Now if we unlabel these groups, the order of group 1 to 4 do not matter, and any order of group 5 and 6 do not matter. This means we have to divide by 3! for groups 1 to 4, and by 2! for group 5 and 6. So our answer for unlabelled groups is now:

$$\frac{25!}{(3!)^4 (4!)^2 5!} \times \frac{1}{4!2!}$$

Exercise 1.4 Find the number of integer solutions to the equation

$$a + b + c + d + e = 120$$

where $a \geq 6$, $b \geq -5$, $c \geq 11$, $d \geq -7$, and $e \geq 8$. What if also require $b \leq 9$? ■

First let's relabel. Define the following:

$$\begin{aligned} a' &= a - 6 \geq 0 \\ b' &= b + 5 \geq 0 \\ c' &= c - 11 \geq 0 \\ d' &= d + 7 \geq 0 \\ e' &= e - 8 \geq 0 \end{aligned}$$

so this gives:

$$\begin{aligned} a'+b'+c'+d'+e' &= (a-6)+(b+5)+(c-11)+(d+7)+(e-8) \\ &= (a+b+c+d+e)-13 \\ &= 120-13 \\ &= 107 \end{aligned}$$

this means we are gifted with a whopping 107 stars, and 4 bars, which we will not diagram for sake of clarity. This means our total count of integer solutions is given by

$$\binom{\text{stars + bars}}{\text{bars}} = \binom{111}{4} = \frac{111!}{107! \times 4!}$$

Now suppose we add the restriction that $b \leq 9$. We will first find the number of solutions given $b > 9$, or $b \geq 10$. In this case, using the same substitutions as before, in this case we let $\beta = b - 10 \geq 0$. So we have:

$$a' + \beta + c' + d' + e' = 120 - 28 = 92$$

as before, our total number of integer solutions to this is $\binom{96}{4}$. Subtracting this from our pervious answer gives:

$$\binom{111}{4} - \binom{96}{4}$$

Exercise 1.5 Give a combinatorial proof of each of the following identities, by recognizing each side as counting the same thing:

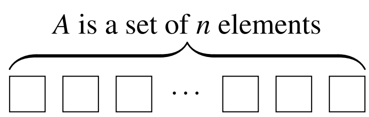
1. $$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$$
2. $$1+2+3+\cdots+(n-1) = \binom{n}{2}$$
3. $$\sum_{k=m}^{n} \frac{k!}{(k-m)!}\binom{n}{k} = \frac{n!}{(n-m)!}2^{n-m}$$

■

Let's start with the first one. Suppose we have a set A with m elements, and B with n elements. Then the left side is telling us how many subsets of size 2 we can make from $A \cup B$. We can break this up into 3 disjoint partitions.

- Subsets of size two from just group A.
- Subsets of size two from just group B
- Subsets of size two, with one element from A, and one element from B.

The first is simply $\binom{m}{2}$, the second being $\binom{n}{2}$, and the last partition being size mn. Adding these together, the left side, we get $\binom{m+n}{2}$ which confirms the equation. On to (2), the right side tells us how many subsets of two exist from A with $|A| = n$. We can divide this into $(n-1)$ partitions. Say we have



Take the first element, then there are $n-1$ possible other elements you can add to this two-set. Then how many unique sets are there with the second element? There are now only $n-2$ possible elements. This continues, and by the time we have the $(n-1)$th element, there is only one more element we can add, the nth element. This corresponds to the left side. Now (3) is a little bit tricky so we will look at the left and right side separately, and show they count the same thing.

- **Left side.** First set m as a constant. Suppose we have a set A with $|A| = n$. Then $\binom{n}{k}$ chooses a subset of size k from A. Multiplying by $\frac{k!}{(k-m)!}$ chooses m *unique* objects from this subset. We need to sum from $k = m$ for it to make sense to choose m unique objects, all the way to n which is the maximum subset we are allowed. *This will give us the total subsets of A with m uniquely marked elements.*
- **Right side.** Now we will work in reverse. The term $\frac{n!}{(n-m)!}$ chooses m unique elements from A. Then we multiply by 2^{n-m} to decide what remaining elements to put into the subsets. *This will give us the total subsets of A with m uniquely marked elements.*

Exercise 1.6 Use algebraic or combinatorial techniques to evaluate the following sums:

1. $$\sum_{k=1}^{n} 5^k k \binom{n}{k}$$
2. $$\sum_{k=0}^{n} \frac{(-3)^k}{k+1} \binom{n}{k}$$
3. $$\sum_{k=0}^{n} k^2 \binom{n}{k}$$

■

For (1), we can apply binomial theorem with $y = 1$ to get

$$\begin{aligned}
(x+1)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \\
\frac{d}{dx}(x+1)^n &= \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} x^k \\
n(x+1)^{n-1} &= \sum_{k=0}^{n} k x^{k-1} \binom{n}{k} \\
nx(x+1)^{n-1} &= \sum_{k=0}^{n} k x^k \binom{n}{k} \\
&= \sum_{k=1}^{n} k x^k \binom{n}{k}
\end{aligned}$$

and letting $x = 5$ gives us

$$\sum_{k=1}^{n} 5^k k \binom{n}{k} = \sum_{k=1}^{n} 5^k k \binom{n}{k} = 5n \cdot 6^{n-1}$$

Now, for (2) we let $y = 1$ in the binomial theorem to get

$$
\begin{aligned}
(x+1)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \\
\int (x+1)^n \, dx &= \int \sum_{k=0}^{n} \binom{n}{k} x^k \, dx \\
\frac{(x+1)^{n+1}}{n+1} &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} x^{k+1} + C
\end{aligned}
$$

now to find C let $x = 0$ to see

$$
\begin{aligned}
\frac{(0+1)^{n+1}}{n+1} &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} 0^{k+1} + C \\
\frac{1}{n+1} &= C
\end{aligned}
$$

so substituting this back in, letting $x = -3$ and dividing by -3 gives

$$
\begin{aligned}
\frac{(x+1)^{n+1}}{n+1} + \frac{1}{n+1} &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} x^{k+1} \\
\frac{(x+1)^{n+1}+1}{n+1} &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} x^{k+1} \\
\frac{(-2)^{n+1}+1}{n+1} &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} (-3)^{k+1} \\
\frac{(-2)^{n+1}+1}{-3(n+1)} &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} (-3)^{k}
\end{aligned}
$$

R Notice we divide by -3 instead of x, and we do this *after* subsituting $x = 0$ to find C. This is to avoid division by 0.

Lastly we will work on (3). Suppose we have a set S with n elements. Then $\binom{n}{k}$ chooses a subset of S of size k. Multiplying this by k is designating a specific element of that subset, and multiplying by k again we are designating another member of this subset, which could very well be the same. Summing these up gives the total subsets of S with two marked elements. To do this in the reverse order, lets look by case:

1. **The two marked objects are the same:** In this case, there are n choices for the double marked object. Then multiply this by 2^{n-1} to decide what objects to add to this subset.
2. **The two marked objects are different:** In this case, there are $n(n-1)$ ways to choose the first marked, and second marked, unique objects. Multiply this by 2^{n-2} to choose what other elements to add to this subset.

Adding these disjoint cases together give:

$$
\begin{aligned}
n2^{n-1} + n(n-1)2^{n-2} &= n2^{n-2}(2 + (n-1)) \\
&= n(n+1)2^{n-2}
\end{aligned}
$$

Exercise 1.7 Given that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

for all x, prove the identity $e^x e^{-y} = e^{x-y}$. ■

We start by looking at the product:

$$e^x e^{-y} = \left[\sum_{j=0}^{\infty} \frac{x^j}{j!}\right]\left[\sum_{k=0}^{\infty} \frac{(-y)^k}{k!}\right]$$

then gathering terms with the total same degree, and setting $r = j+k$ we get

$$e^x e^{-y} = \sum_{r=0}^{\infty}\sum_{k=0}^{r} \frac{x^k(-y)^{r-k}}{k!(r-k)!}$$

now we want to end up with an expression that has $\frac{1}{r!}$ in it, so it's a good idea to factor that out. We can do that by seeing:

$$\begin{aligned} \binom{r}{k} &= \frac{r!}{k!(r-k)!} \\ \frac{1}{r!}\binom{r}{k} &= \frac{1}{k!(r-k)!} \end{aligned}$$

thus

$$\begin{aligned} e^x e^{-y} &= \sum_{r=0}^{\infty}\sum_{k=0}^{r} \frac{x^k(-y)^{r-k}}{k!(r-k)!} \\ &= \sum_{r=0}^{\infty}\sum_{k=0}^{r} \frac{1}{r!}\binom{r}{k} x^k(-y)^{r-k} \\ &= \sum_{r=0}^{\infty}\frac{1}{r!}\sum_{k=0}^{r} \binom{r}{k} x^k(-y)^{r-k} \\ &= \sum_{r=0}^{\infty} \frac{(x-y)^r}{r!} \\ &= e^{x-y} \end{aligned}$$

R Technically on that step where we wrote $e^x e^{-y} = \sum_{r=0}^{\infty}\sum_{k=0}^{r} \frac{x^k(-y)^{r-k}}{k!(r-k)!}$ the labeling of the exponents has been swapped, to match the binomial theorem given in the notes. This results in the same answer anyway.

2. Matchings and Polynomials

I'll *match* that!

Monty Hall

2.1 Structure

We begin with several applications of the pigeonhole principle, including repeated dice outcomes, equal-sum subsets, and guaranteed goal totals over consecutive games. It then shifts to matchings and transversals in bipartite graphs using Hall's theorem (including an application to subset containment graphs). The final problems use inclusion-exclusion to count bounded integer solutions and rook polynomials to enumerate rook placements and perfect matchings.

2.2 Exercises

Exercise 2.1 Marama is conducting an experiment on rolling dice for her statistics course. She has three ordinary six-sided dice: the first is red, the second is black, and the third is white. One trial of the experiment consists of rolling the three dice and recording the sequence of three numbers visible uppermost, in order red, black, white. She then also calculates the sum of the three numbers.

1. If Marama conducts a total of 1000 trials, what's
   (a) the largest value of k for which we can guarantee that at least k of the trials will have the same sequence of three numbers?
   (b) the largest value of k for which we can guarantee that at least k of the trials will have the same sum?
   (c) the smallest value of k for which we can guarantee that no more than k of the trials have the same sum?
2. What's the smallest number of trials Marama must conduct in order to guarantee that at least 50 of them have the same sum?

■

For the first question, we are given 3 *distinct* dice. First, since there are 6 sides of each die, there are $6^3 = 216$ possible sequence of rolls. With 1000 rolls, even if we somehow got all 216 possible sequences the first 216 tries (highly unlikely), we would be forced to have another roll that we have already done before. In the end, we can only *guarantee* at least

$$k = \left\lceil \frac{\text{Pigeons}}{\text{Pigeon Boxes}} \right\rceil = \left\lceil \frac{1000}{216} \right\rceil = 5$$

trials share the same sequence.
As for the second part of (1), we will first find how many possible sums there are. The smallest possible sum we could get is 3, given by a roll of 3 ones. The largest possible sum we could get is 18. Therefore there are $15+1=16$ possible sums. Again, even if we tried to spread out all 1000 trials, over 16 possible sums, we would have at least

$$k = \left\lceil \frac{1000}{16} \right\rceil = 63$$

trials that share the same sum.
Now for the last part of (1), simply choose $k = 1000$. Suppose (highly unlikely) that all the 1000 trials have the same sum, then *we can guarantee no more than 1000 of these trials have the same sum*.

Now for (2) recall there are 16 possible sums. Suppose we roll the dice 17 times, then by pigeonhole principle we must have at least 2 trials with the same sum. Suppose we roll 33 times. Then we must have at least 3 trials having the same sum. Extending this logic, if we roll $(16\times 49)+1 = 785$ times, we must have at least 50 trials having the same sum (best case scenario).

Exercise 2.2 Let $X = \{6,7,8,...,60\}$, and let S be a subset of X of size 10. Use the pigeonhole principle to show that there must be two subsets of S of size 5 that have the same sum. ■

Again, when using this pigeonhole principle we want to find our boxes, find our pigeons, and show that there exist more pigeons than boxes. So we have our set S that has 10 elements. The number of possible subsets of S, of size 5 is given by $\binom{10}{5} = 252$. These possible subsets are our pigeons. Our boxes are the possible sums of any subsets of S. The maximum sum is given by $60+59+58+57+56 = 290$. The minimum possible sum is given by $6+7+8+9+10 = 40$. Therefore our possible sums could be anywhere from 40 to 290, giving us *at most* $250+1 = 251$ boxes. Since there are more boxes than pigeons, by pigeonhole principle, there must exist at least one box with two pigeons in it. Since $252 > 251$, two subsets of size 5 must share the same sum.

Exercise 2.3 Three soccer teams — the Ants, the Bees, and the Cicadas — each played 40 games this year, and each scored at least one goal per game.
1. The Ants scored a total of 71 goals. Show that there must be a sequence of consecutive games in which they scored exactly 8 goals.
2. The Bees scored a total of 80 goals. Give an example to show that there doesn't have to be a sequence of consecutive games in which they scored exactly 8 goals.
3. The Cicadas scored a total of 79 goals. Prove or disprove: There must be a sequence of consecutive games in which they scored exactly 8 goals.

■

Let G_i denote the number of goals scored in the $i-$th game. Here $i \in \{0,\dots,40\}$, and $\sum_i G_i = 71$, including the restriction $G_i \geq 1$ for all i, and $G_0 = 0$. The question is asking about a sequence of consecutive games, so we need to keep track of the total goals scored at each game i. This motivates the definition of $T_n := \sum_{k=0}^{n} G_i$ being the total goals scored up to G_n. See that $G_{40} = 71$. What we need to show is that there must be some sequence in which 8 goals were scored, or equivalently, there must exist i and j with $i \leq j$ such that $T_j = T_i + 8$. Consider the 82 elements $T_0, T_1, \dots, T_{40}$

and $T_0+8, T_1+8, \dots, T_{40}+8$. The largest value for any of these is given by $T_{40}+8=79$. The smallest possible value is 0. This means there are a total 80 possible values for the 82 elements. The pigeonhole principle states that there must be two elements with the same value from the set $T \cup T+8$. But remember that any two elements in T must be different, and any two elements in $T+8$ must be different. This means that the elements with the same value from the different sets. This shows there exists T_i and T_j such that $T_i = T_j + 8$, completing the proof.

Now for (2), by inspection we can make the following block. First let $G_1, \dots, G_7 = 1$. This means we have a consecutive run of 1 goals each game, and in total 7 goals won so far. Then let $G_8 = 9$. This ensures two things. Firstly, if this sequence is repeated, no consecutive games will ever be equal to 8, since 9 already oversteps this. Secondly, since we have got a block of 16 wins in 8 games, we simply repeat this 5 times to get a total of 40 won games. Our final sequence is defined as

$$G_i := \begin{cases} 1, & i \text{ is not divisible by } 8 \\ 9, & \text{otherwise} \end{cases}$$

for $i \in \{1, \dots 40\}$.

Moving onto (3). We will show that it is impossible to avoid making a sequence of consecutive games in which 8 goals were scored. The key is to observe the sequence T_n. Suppose we place down the goals in the sequence such that $T_7 = 10$. Then if we wish to avoid a consecutive sum of 8 goals, we cannot have $T_i = 18$ at any term in the future, and we cannot have $T_i = 2$ at any term placed previously. This is because in both cases we have a difference of 8. In this case, because we have organised our sequence such that $T_7 = 10$ we can no longer choose 18 or 2, and similarly, if we chose $T_7 = 18$ we can no longer choose 26 or 10 for any term in the sequence. We can organise this as so:

$$2, 10, 18, 26, 34, 42, 50, 58, 66, 74$$

here the minimum is 2 since the minimum total is $T_0 = 0$. We end at 74 since the largest total is 79. Whenever we choose some number in that list for any term T_i we can no longer choose the number to the left or right of it for any other term in $\{T_n\}$. Of course, we have a whole range of possible T_i to choose from, as seen in the following table:

Sequence	**Numbers in the Sequence**									
Starts with 0	0	8	16	24	32	40	48	56	64	72
Starts with 1	1	9	17	25	33	41	49	57	65	73
Starts with 2	2	10	18	26	34	42	50	58	66	74
Starts with 3	3	11	19	27	35	43	51	59	67	75
Starts with 4	4	12	20	28	36	44	52	60	68	76
Starts with 5	5	13	21	29	37	45	53	61	69	77
Starts with 6	6	14	22	30	38	46	54	62	70	78
Starts with 7	7	15	23	31	39	47	55	63	71	79

The above table lists possible T_ns grouped into sequences with increments of 8. Suppose there were no restrictions, then we are free to choose 41 numbers from this box (but with $T_0 = 0$ and $T_{40} = 79$ forced). But given the restrictions, whenever we choose one number, we have to cancel out the number on the left and right, in that row. Given any row, the absolute maximum number of elements we can choose are 5. For example, in the 5th row we could choose $5, 21, 37, 53, 69$. Because we have 8 rows, there are 40 options we can choose from. The problem is we have to choose 41 numbers from these options. This is not possible, so that extra number must be adjacent to (within 8 of) some other value that has already been selected. This proves that no sequence can be constructed that avoids 8 consecutive goals. Therefore: every sequence will include some consecutive games in which 8 goals are scored.

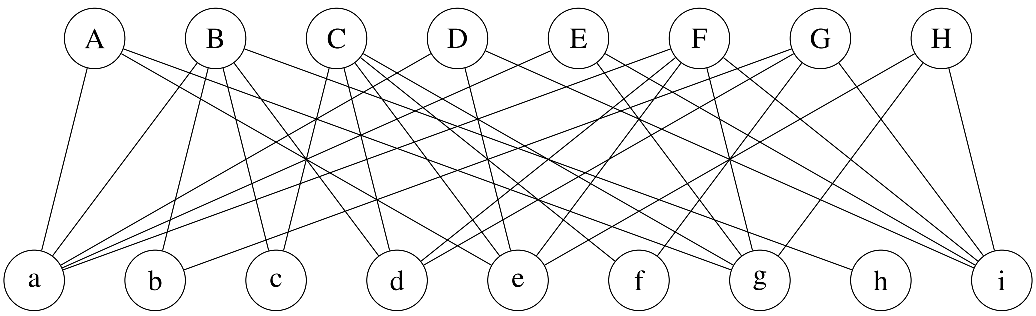
R One can see if a total of 80 goals were scored we would have $T_{40} = 80$. In our table, we would have an extra number to choose from, without having to collide with any other terms. From rows 0-6 we can choose a maximum of 5 numbers; 35 numbers total. But now in the last row we can have 6 choices, giving us a total of 41 options for the terms in the sequence. Since we have to choose 41 terms, it is possible to choose them such that there is no collisions.

Exercise 2.4 Let $\mathscr{X} = (A,B,C,D,E,F,G,H)$ be the following family of sets:

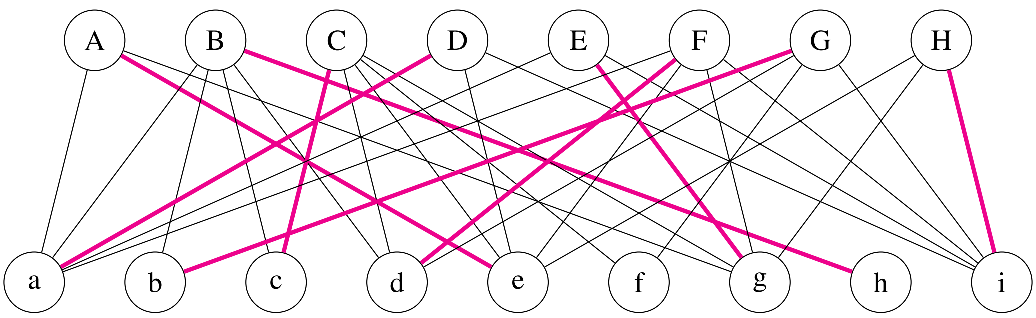
$$A = \{a,e,g\}, \quad C = \{c,d,e,f,g\}, \quad E = \{a,g,i\}, \quad G = \{b,d,f,i\},$$
$$B = \{a,b,c,d,h\}, \quad D = \{a,e,i\}, \quad F = \{a,d,e,g,i\}, \quad H = \{e,g,i\}.$$

1. Draw a bipartite graph $\mathscr{G}$ such that the problem of finding a transversal of $\mathscr{X}$ is equivalent to the problem of finding a matching covering $\{A,B,C,D,E,F,G,H\}$ in $\mathscr{G}$.
2. Find a transversal of $\mathscr{X}$.
3. Find a set $S \in \{A,B,C,D,E,F,G,H\}$, and an element $s \in S$, such that deleting s from S (that is, replacing S with $S' = S - \{s\}$, and leaving the other sets unchanged) gives us a new family $\mathscr{X}'$ that does not have a transversal.
   This is equivalent to finding an edge $t = (S,s)$ in the graph $\mathscr{G}$ such that deleting t from $\mathscr{G}$ gives us a new graph $\mathscr{G}'$ that does not have a matching covering $\{A,B,C,D,E,F,G,H\}$. ■

For (1) we draw the graph as so:



We highlight a transversal as follows:



Indeed Pollock does it better. You can check this is transversal since each uppercase letter is mapped to a unique lowercase letter.

Theorem 2.2.1 — Hall. Let G be a bipartite graph with bipartition $\{A, B\}$. Then A can be matched into B if an only if $|N(J)| \geq |J|$ for all subsets $J \subseteq A$.

Now we move on to (3). Here we should start by finding the smallest sets. These would be A, D, E, and H. Notice their union is $\{a, g, i, e\}$. Suppose look at F. It has one more element, that is not included in the other sets mentioned, namely d. This motivates us to choose $S = F$, with $F' = F - d$. This means we have 5 sets that have to map to 4 elements, which means there cannot exist any bijective map between them. This is a violation of Hall's theorem as:

$$|A \cup D \cup E \cup H| = 5 < 4$$

Exercise 2.5 This question has two parts:

1. Let G be a bipartite graph with bipartition $\{A, B\}$. Suppose that every vertex in A has degree α and every vertex in B has degree β, where α and β are positive integers such that $\alpha \geq \beta$. Prove that G has a matching which covers A.
2. Let k and n be positive integers such that $n \geq 2$ and $k \leq n/2$. Let A be the set of subsets of $\{1, 2, \ldots, n\}$ of size $k-1$; let B be the set of subsets of $\{1, 2, \ldots, n\}$ of size $k+1$; and let G be the bipartite graph with bipartition $\{A, B\}$, where we draw an edge from $a \in A$ to $b \in B$ if $a \subseteq b$.
   Use (a) to prove that G has a matching which covers A.

■

For (1) we will aim to use Hall's theorem. In other words, we need to show that for any collection $J \subseteq A$, then $|N(J)| \geq |J|$, where $N(J) \subseteq B$ represents the verticies J is sent to (set of edges). Suppose we choose some set $J \subseteq A$. Then the number of verticies in J is denoted $|J|$. This means the number of edges leaving J is given by $\alpha|J|$. Similarily, if $N(J)$ is the number of verticies mapped to by J, then $\beta|N(J)|$ is the number of edges leaving $N(J)$. By definition of $N(J)$, the number of edges leaving J has to be less than or equal to the number of edges arriving in $N(J)$. This is becuase the verticies in $N(J)$ could be getting more edges from other verticies in $A - J$. So in summary for any $J \subseteq A$ we have

$$\begin{aligned} \alpha|J| &\leq \beta|N(J)| \\ |J| \cdot \alpha/\beta &\leq |N(J)| \\ |J| &\leq |N(J)| \end{aligned}$$

the last line follows since $\alpha/\beta \geq 1$. By theorem 2.2.1 there must exist a matching that covers A. Now for (2), we are given a hint we need to use (1). We can show that a matching can be found between A and B if we can find the degree of the verticies. Suppose we have a vertex $a \in A$. Then $|a| = k-1$ by definition. We want to see how many verticies in B are mapped to by a. Recall a is only mapped to b if $a \subseteq b$. Recalling that $|b| = k+1$, we choose two elements from the $n - (k-1)$ possible elements. So, there are $\binom{n-k+1}{2}$ neighbors of $a \in B$. Therefore $\deg(a) = \binom{n-k+1}{2}$ for all $a \in A$. Now lets take some $b \in B$. Then b is connected to any element in $a \in A$ only if $a \subseteq b$. Since b has $k+1$ elements, there are $\binom{k+1}{2}$ possible ways to choose 2 elements from b, to delete. Thus, $\deg(b) = \binom{k+1}{2}$. This is true for all $b \in B$. All that remains is to show $\deg(a) \geq \deg(b)$. Recall

$k \leq n/2$. Then

$$\begin{aligned} k &\leq n/2 \\ 2k &\leq n \\ k &\leq n-k \\ k+1 &\leq n-k+1 \\ \binom{k+1}{2} &\leq \binom{n-k+1}{2} \\ \deg(b) &\leq \deg(a) \end{aligned}$$

which by (1) shows there must exist a matching which covers A.

Exercise 2.6 Use the principle of inclusion-exclusion to find the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 180$$

such that

1. $x_i \leq 30$ for all i.
2. $x_i > 30$ for exactly 3 indicies i.

■

We first recall the principle of inclusion-exclusion, for compliments of sets.

Theorem 2.2.2 — Inclusion-Exclusion for Complements. Given subsets $A_1, \ldots, A_n$ of a finite set S, then

$$|S - (A_1 \cup \cdots \cup A_n)| = |S| + \sum_{k=1}^{n} (-1)^k F(k)$$

where $F(k)$ is defined as

$$F(k) = \sum_{i_1 < i_2 < \cdots < i_k} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

Let's first denote A_i as the set of solutions to the equation, with $x_i \geq 31$. Let's consider restircting only one element (say the first), as $x_1 \geq 31$. Then the number of solutions to this is given by $\binom{186-31}{6}$. Now instead of just considering only $x_1 \geq 1$, there are 6 other options. So in summary we have $F(1) = 7\binom{155}{6}$. Now suppose we restrict two arbitrary elements, in this case x_1 and x_2 above or equal to 31. Then the number of solutions to the equation is given by $\binom{186-62}{6}$. But there are $\binom{7}{2}$ ways to choose those two elements, thus $F(2) = \binom{7}{2}\binom{124}{6}$. This pattern can be generalised as

$$F(k) = \binom{7}{k}\binom{186-31k}{6}$$

obviously this vanishes when $k = 6$ since the second binomial becomes 0. Now our total space S is the set of all non-ngegative solutions to the equation, with no restrictions. In this case $|S| = \binom{186}{6}$. So we calculate:

$$\begin{aligned} |S - (A_1 \cup \cdots \cup A_n)| &= |S| + \sum_{k=1}^{7} (-1)^k F(k) \\ &= |S| - F(1) + F(2) - F(3) + F(4) - F(5) + F(6) - F(7) \\ &= \binom{186}{6} - \binom{7}{1}\binom{155}{6} + \binom{7}{2}\binom{124}{6} - \binom{7}{3}\binom{93}{6} + \binom{7}{4}\binom{62}{6} - \binom{7}{5}\binom{31}{6} \end{aligned}$$

The above when calculated will give the number of non-negative, under or equal to 30, integer solutions to the equation.
Now for (2) we need to recap the definition of inclusion exclusion, in full.

Theorem 2.2.3 — Inclusion-Exclusion. Let S be a finite set, and $A_1,\dots,A_n$ be subsets of S. Then for $0 \le m \le n$, let

$$X_m := \{x \in S : x \text{ belongs to exactly } m \text{ of } A_1,\dots,A_n\}$$

Then

$$|X_m| = \sum_{k=m}^{n} (-1)^{k+m} \binom{k}{m} F(k)$$

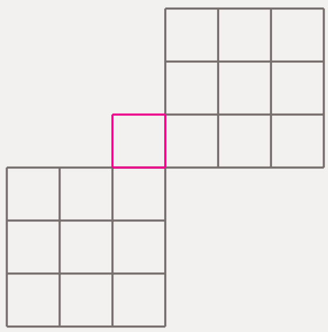
where $F(0) = |S|$.

Here we need to find X_3. So $m = 3$ and $n = 7$. We have already found our expressions for $K(1),\dots,K(7)$. So we wish to find

$$\begin{aligned}
|X_3| &= \sum_{k=3}^{7} (-1)^k + 3\binom{k}{3} F(k) \\
&= \binom{3}{3}F(3) - \binom{4}{3}F(4) + \binom{5}{3}F(5) - \binom{6}{3}F(6) + \binom{7}{3}F(7) \\
&= \binom{7}{3}\binom{93}{6} - 4\binom{7}{4}\binom{62}{6} + 10\binom{7}{5}\binom{31}{6}
\end{aligned}$$

and this answers (2).

Exercise 2.7 Find with proof the rook polynomial of the following board:



■

Let's do this by divide and conquer, for this board we label B. In order to carry this out we will need two tools. First is a formal statement of divide and conquer. The second are some rook polynomials of some boards we will need to use later.

Theorem 2.2.4 — Divide and Conquer. Given a board B, with some square $S \in B$, we denote S° as the set of squares that make up the row and columns that contain S. Then

$$R(x,B) = xR(x, B - S^\circ) + R(x, B - S)$$

Proposition 2.2.5 The rook polynomial of a 2×3 board is given by

$$R(x, B_{2\times 3}) = 1 + 6x + 6x^2$$

and for a 3×3 board we have

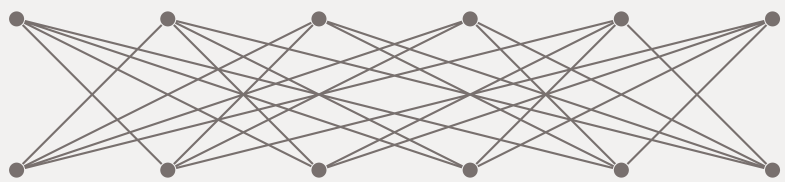
$$R(x, B_{3\times3}) = 1 + 9x + 18x^2 + 6x^3$$

Proof. For the first equation, see that r_0 is always 1. Then r_1 is always how many squares there are on the board (in this case 6). For r_2,see that there are 3 possible columns to put two rooks in. So there are $\binom{3}{2} = 3$ possible ways to choose the columns. Once the columns have been chosen, there are only 2 possible ways to arrange the rooks via row, to avoid collision. So we have $r_2 = 6$.
For the second equation, we also know that $r_0 = 1$ and $r_1 = 9$. As in the previous explanation, there are $\binom{3}{2} = 3$ possible ways to choose 2 columns from 3. Then there are $\binom{3}{2} = 3$ was to choose the rows these rooks sit. This would leave us with 4 squares, in which there are 2 ways to arrange these rooks. So $r_2 = 3 \times 3 \times 2 = 18$. Now r_3 is trivial since there is only one way to choose 3 columns and 3 rows. One can see there are $3! = 6$ ways to arrange these rooks such that they do not collide. So $r_3 = 6$. ■

We now have enough to solve this problem. Let S be the magenta square that connects the two 3×3 boards. Then we have

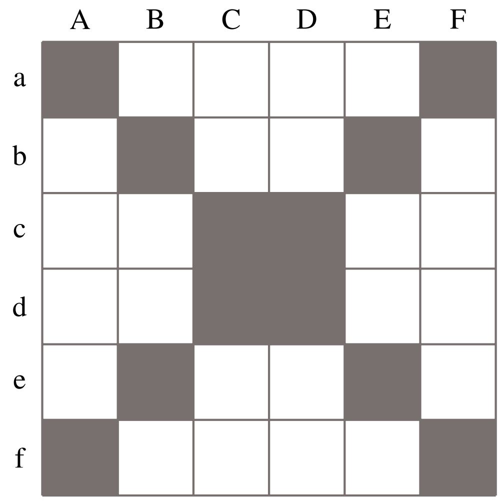
$$\begin{aligned}
R(x, B - S^\circ) &= R(x, B_{2\times3} \cup B_{2\times3}) \\
&= R(x, B_{2\times3})^2 \\
&= (1 + 6x + 6x^2)^2 \\
R(x, B - S) &= R(x, B_{3\times3} \cup B_{3\times3}) \\
&= R(x, B_{3\times3})^2 \\
&= (1 + 9x + 18x^2 + 6^3)^2 \\
\therefore \quad R(x, B) &= xR(x, B - S^\circ) + R(x, B - S) \\
&= xR(x, B_{2\times3} \cup B_{2\times3}) + R(x, B_{3\times3} \cup B_{3\times3}) \\
&= x(1 + 6x + 6x^2)^2 + (1 + 9x + 18x^2 + 6^3)^2 \\
&= 47089 + 3907x + 7905x^2 + 372x^3 + 396x^4 + 36x^5
\end{aligned}$$

Exercise 2.8 Use the rook polynomial of the "board of forbidden positions" to determine the number of perfect matchings in the graph below:



■

We first will label the top verticies as A, B, C, D, E, F and the bottom verticies as a, b, c, d, e, f. We will then draw a grid with the x-axis being the top row of verticies, and the y-axis presenting the bottom row of verticies. We then fill in the (i, j)-squares in which letter i isn't conencted to letter j. This symbolises there is no edge between the two verticies, so we will have a "board of forbidden positions":



While we could try to figure out the rook polynomial of the white squared board, its much easier to calculate this for the dark squares. Remember that while the corners $(A,a),(A,f),(F,a)$, and (F,f) look disjoint, they actually form their own 2×2 board since they share the same column and rows. So we have 3 disjoint 3×3 boards. Therefore the rook polynomial of this dark board in total given by

$$R(x,B) = (1+4x+2x^2)^3 = 1+12x+54x^2+112x^3+108x^4+48x^5+8x^6$$

so we are able to find the rook polynomial of the white squares is given by

$$\begin{aligned} r_7(\text{White part}) &= \sum_{k=0}^{6}(-1)^k(7-k)!r_k(\text{Dark part}) \\ &= (6!)(1)-(5!)(12)+(4!)(54)-(3!)(112)+(2!)(108)-(1!)(48)+(0!)(8) \\ &= (720)(1)-(120)(12)+(24)(54)-(6)(112)+(2)(108)-(1)(48)+(1)(8) \\ &= 80 \end{aligned}$$

which means there are 80 possible perfect matchings in the given graph.

3. Generating Functions

It's *clearly* a budget. It's got a lot of numbers in it.

George W. Bush

3.1 Structure

This chapter has an emphasis on generating functions and recurrence methods. It begins by building generating functions for constrained integer solutions and for partition families, then uses recurrences (via characteristic equations and determinants) to obtain explicit formulas. The later exercises develop counting recurrences for restricted digit strings and use generating functions to solve non-homogeneous recurrences and constrained composition counts. The chapter concludes with an application of Burnside's theorem to count inequivalent colourings of a symmetric stained-glass octagon under rotations and reflections.

3.2 Exercises

Exercise 3.1 The equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = n$$

is to be solved in non-negative integers. Find the generating function for the number of solutions subject to the following constraints:

- $x_1 \leq 5$;
- $x_2 \geq 3$;
- no restrictions on x_3;
- x_4 is a multiple of 4;
- $x_5 \in \{2, 3, 5, 7\}$;
- and $x_6 = x_3 + 3$

■

To look at this we will find the generating function $G_i(r)$ of each variable x_i with the given restriction, then multiply them together to get the total solutions to the equation. Let's do this first for $i = \{1,2,4,5\}$. For $x_1 \leq 5$ we have

$$G_1(r) = 1 + r + r^2 + r^3 + r^4 + r^5 = \frac{1-r^6}{1-r}$$

Since we have $x_2 \geq 3$ we multiply through by r^3 to get

$$G_2(r) = r^3 + r^4 + r^5 + r^6 + \cdots = \frac{r^3}{1-r}$$

Skipping 3 lets look at x_4. This looks like

$$G_4(r) = 1 + r^4 + r^8 + r^{12} + r^{16} + \cdots = \frac{1}{1-r^4}$$

Lastly it is best to leave

$$G_5(r) = r^2 + r^3 + r^5 + r^7$$

Now the problem we face is that x_6 is dependant on x_3. Thankfully this dependance is something we can model. Since there are no restrictions on x_3 this can range from $[0,\infty) \subseteq \mathbb{Z}$. Since$x_6 = x_3 + 3$ we have $x_3 + x_6 \in \{3,5,7,9,\ldots\}$. This is because if $x_3 = 0$ then $x_6 = 3$ so $x_3 + x_6 = 3 + 0 = 3$, and so on. Therefore our generating function of $x_3 + x_6$ is given by

$$G_{3\text{ and }6}(r) = r^3 + r^5 + r^7 + r^9 + \cdots = \frac{r^3}{1-r^2}$$

In the end we are able to multiply all these to get our final answer:

$$G_1(r)G_2(r)G_4(r)G_5(r)G_{3\text{ and }6}(r) = \frac{1-r^6}{1-r}\frac{r^3}{1-r}\frac{1}{1-r^4}\frac{(r^2+r^3+r^5+r^7)r^3}{(1-r^2)}$$

This can be simplified, so the generating function for the number of non-negative integer solutions to the equation, subject to said restraints is:

$$\frac{(r^6-r^{12})(r^2+r^3+r^5+r^7)}{(1-r)^2(1-r^4)(1-r^2)}$$

Exercise 3.2 This problem has 5 sub parts.

1. Write down the generating function for the number of partitions of n where no part occurs more than three times.
2. Write down the generating function for the number of partitions of n where no part is divisible by 4.
3. Write down the generating function for the number of partitions of n where only odd parts may be repeated (this implicitly means that all even parts in the partition are distinct).
4. Use algebraic manipulations to show that the generating functions you found in (1)–(3) above are all equal. Hence conclude that the numbers of partitions of n as in each of (1), (2) and (3) are equal for all n.
5. There are 22 partitions of 8. List them, and use your list to verify the conclusion of (3) in the case $n = 8$.

■

For (1) by forcing each part to go up to 3 we are making each factor $1 + x^i + x^{2i} + x^{3i}$. So for the partitions of n we get

$$G_1(x) = \prod_{i=1}^{\infty}(1 + x^i + x^{2i} + x^{3i})$$

Now see this is equivelant to

$$G_1(x)=\prod_{i=1}^{\infty}\frac{1-x^{4i}}{1-x^i}$$

For (2) we need to get rid of any factors divisible by 4. To do this, we will take the unrestricted generating function $\prod_{i=1}^{\infty}\frac{1}{1-x^i}$ and multiply this by each $1-x^{4i}$. This has the intended effect of cancelling out any $\frac{1}{1-x^{4i}}$ terms in the original generating function. So we have

$$G_2(x)=\left(\prod_{i=1}^{\infty}\frac{1}{1-x^i}\right)\left(\prod_{i=1}^{\infty}1-x^{4i}\right)=\prod_{i=1}^{\infty}\frac{1-x^{4i}}{1-x^i}$$

notice this is equal to $G_1(x)$.
Moving on to (3). Let i be odd, then since it can be chosen any number of times we have the standard $\sum_{i=1}^{\infty}\frac{1}{1-x^i}$ as the first factor. Suppose i is even, then we can only allow $\prod_{i=1}^{\infty}1+x^i$ as a factor. We can't multiply these just yet becuase i in the first one is odd, and i in the second one is even. To fix this let $k=1$ run to infinity, so we can multiply like so:

$$G_3(x)=\left(\prod_{k=1}^{\infty}\frac{1}{1-x^{2k-1}}\right)\left(\prod_{k=1}^{\infty}(1+x^{2k})\right)=\left(\prod_{k=1}^{\infty}\frac{1}{1-x^{2k-1}}\right)\left(\prod_{k=1}^{\infty}\frac{1-x^{4k}}{1-x^{2k}}\right)=\prod_{k=1}^{\infty}\frac{1-x^{4k}}{(1-x^{2k-1})(1-x^{2k})}$$

and since $(1-x^2)=(1+x)(1-x)$ and thus $1-x^{4k}=(1+x^{2k})(1-x^{2k})$ we have (after relabelling $i\iff k$)

$$G_3(x)=\prod_{i=1}^{\infty}\frac{(1+x^{2i})(1-x^{2i})}{(1-x^{2i-1})(1-x^{2i})}=\prod_{i=1}^{\infty}\frac{1+x^{2i}}{1-x^{2i-1}}=\prod_{i=1}^{\infty}\frac{1+x^{2i}}{1-x^{2i-1}}\cdot\frac{1-x^{2i}}{1-x^{2i}}=\prod_{i=1}^{\infty}\frac{1-x^{4i}}{1-x^i}$$

Notice this is equal to $G_1(x)$.

R I just realised we can skip a step, that is to show:

$$\prod_{i=1}^{\infty}\frac{1-x^{4i}}{(1-x^{2i-1})(1-x^{2i})}$$

is the same as

$$\prod_{i=1}^{\infty}\frac{1-x^{4i}}{1-x^i}$$

This is because the denominator in the first equation has one factor which takes in every odd number, and another one which takes every even number. Since these go over every integer once, taking the product of that denominator is the same as taking the product over all integers.

See that we have have shown $G_1(x)=G_2(x)$. Also $G_3(x)=G_1(x)$, and therefore $G_3(x)=G_2(x)$. In summary $G_1(x)=G_2(x)=G_3(x)$ proving (3).
Here is a table which shows all 22 partitions of 8. It sorts which partitions have no part that occurs more than 3 times, no part is divisible by 4, only odd parts may be repeated. Indeed, in each case

there is only 16 partitions that satisfy the requirement.

No part > 3 times	No part divisible by 4	Only odd parts may repeat
8	8	8
7+1	7+1	7+1
6+2	6+2	6+2
6+1+1	6+1+1	6+1+1
5+3	5+3	5+3
5+2+1	5+2+1	5+2+1
5+1+1+1	5+1+1+1	5+1+1+1
4+4	4+4	4+4
4+3+1	4+3+1	4+3+1
4+2+2	4+2+2	4+2+2
4+2+1+1	4+2+1+1	4+2+1+1
4+1+1+1+1	4+1+1+1+1	4+1+1+1+1
3+3+2	3+3+2	3+3+2
3+3+1+1	3+3+1+1	3+3+1+1
3+2+2+1	3+2+2+1	3+2+2+1
3+2+1+1+1	3+2+1+1+1	3+2+1+1+1
3+1+1+1+1+1	3+1+1+1+1+1	3+1+1+1+1+1
2+2+2+2	2+2+2+2	2+2+2+2
2+2+2+1+1	2+2+2+1+1	2+2+2+1+1
2+2+1+1+1+1	2+2+1+1+1+1	2+2+1+1+1+1
2+1+1+1+1+1+1	2+1+1+1+1+1+1	2+1+1+1+1+1+1
1+1+1+1+1+1+1+1	1+1+1+1+1+1+1+1	1+1+1+1+1+1+1+1

Exercise 3.3 Let $k \in \mathbb{R}$. For $n \geq 1$ let d_n be the $n \times n$ determinant

$$d_n = \det \begin{bmatrix} k-1 & k & 0 & 0 & \cdots & 0 \\ -1 & k-1 & k & 0 & \cdots & 0 \\ 0 & -1 & k-1 & k & \cdots & 0 \\ 0 & 0 & -1 & k-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & k-1 \end{bmatrix}$$

Calculate d_n by finding and solving a recurrence relation. ■

Let's try to calculate the first few terms. We wont need to use all them (only need d_1 and d_2):

$$d_0 = 1$$
$$d_1 = \det\begin{bmatrix}k-1\end{bmatrix} = k-1$$
$$d_2 = \det\begin{bmatrix}k-1 & k \\ -1 & k-1\end{bmatrix} = (k-1)(k-1)+k = (k-1)d_1 + kd_0$$
$$\begin{aligned} d_3 &= \det\begin{bmatrix}k-1 & k & 0 \\ -1 & k-1 & k \\ 0 & -1 & k-1\end{bmatrix} = (k-1)\begin{vmatrix}k-1 & k \\ -1 & k-1\end{vmatrix} - k\begin{vmatrix}-1 & k \\ 0 & k-1\end{vmatrix} + \cdots \text{all these terms vanish} \\ &= (k-1)\left[(k-1)^2+(k)\right]+k(k-1) \\ &= (k-1)d_2 + kd_1 \\ &\vdots \end{aligned}$$

So we are able to manipulate the recurrence:

$$\begin{aligned} d_n &= (k-1)d_{n-1} + kd_{n-2} \\ d_n - (k-1)d_{n-1} - kd_{n-2} &= 0 \end{aligned}$$

with characteristic equation:

$$\begin{aligned} x^2-(k-1)x-k &= 0 \\ x^2-(k-1)x &= k \\ x^2-(k-1)x+\frac{(k-1)^2}{4} &= \frac{4k+(k-1)^2}{4} \\ x-\frac{k-1}{2} &= \pm\frac{\sqrt{4k+k^2-2k+1}}{2} \\ x &= \frac{k-1}{2} \pm \frac{k+1}{2} \\ x &= \frac{k-1\pm(k+1)}{2} \end{aligned}$$

which gives roots $x = k$ and $x = -1$. Our general form of the solution is

$$d_n = Ak^n + B(-1)^n$$

Using our initial condition $d_1 = k-1$ we have $Ak - B = k-1$. Since $A+B = 1$, we have $(1-B)k - B = k-1$ so $Bk + B = 1$ and this $B = 1/(k+1)$. This means $A = k/(k+1)$. In summary the formula for the recurrence relation is equal to

$$d_n = \frac{k^{n+1}}{k+1} + \frac{(-1)^n}{k+1} = \frac{k^{n+1}+(-1)^n}{k+1}$$

R Remember that general form only holds with distinct roots, so when $k \neq -1$. In this case, the general solution becomes

$$d_n = A(-1)^n + Bn(-1)^n$$

By inspection d_0 tells us $A = 1$. Then $d_1 = k-1 = -2$ says $-2 = -1-B$ so $B = 1$. So in the sepcial case $k = -1$ we have

$$d_n = (-1)^n + n(-1)^n = (-1)^n(n+1)$$

Exercise 3.4 The sequence a_n is defined by the recurrence relation

$$a_{n+1} = 4a_n - a_{n-1}$$

for $n \geq 1$, with initial conditions $a_0 = 1$ and $a_1 = 2$. Find an explicit formula for a_n. ■

The characteristic equation is given by $x^2 - 4x + 1 = 0$ whose roots can be solved for as so:

$$\begin{aligned} x^2 - 4x &= -1 \\ x^2 - 4x + 4 &= 3 \\ (x-2)^2 &= 3 \\ x - 2 &= \pm\sqrt{3} \\ x &= 2 \pm \sqrt{3} \end{aligned}$$

so the solution is going to look something like

$$a_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n$$

and since we are given $a_0 = 1$ and $a_1 = 2$ then we have the solution of equations

$$\begin{aligned} 1 &= A + B \\ 2 &= A(2+\sqrt{3}) + B(2-\sqrt{3}) \end{aligned}$$

The first one can be rearranged as $B = 1 - A$, so we expand:

$$\begin{aligned} A(2+\sqrt{3}) + B(2-\sqrt{3}) &= 2A + \sqrt{3}A + 2B - \sqrt{3}B \\ &= 2(A+B) + \sqrt{3}A - \sqrt{3}B \\ &= 2 + \sqrt{3}A - \sqrt{3}B \\ &= 2 + \sqrt{3}A - \sqrt{3}(1-A) \\ &= 2 + \sqrt{3}(A - 1 + A) \\ &= 2 + \sqrt{3}(2A-1) \end{aligned}$$

Now recall this is equal to 2 so we let

$$\begin{aligned} 2 + \sqrt{3}(2A-1) &= 2 \\ 2A - 1 &= 0 \\ A &= \frac{1}{2} \\ \therefore \quad B &= \frac{1}{2} \end{aligned}$$

so the formula for our recurrence relation is

$$a_n = \frac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2}$$

Exercise 3.5 Let b_n be the number of n-digit strings that can be made using the ten digits 0–9 such that

- any digit immediately after a 0, 1, or 2 must be a 3 or 4; and
- any digit immediately after a 3 must be a 4 or 5.

So for example, for $n = 6$ the strings 047352 and 913486 are allowed, but 781534 and 623755 are not, because the first has a 5 immediately after a 1, and the second has a 7 immediately after a 3.

Find, with proof, a recurrence relation and initial conditions for b_n. Clearly state your initial conditions.

You are not required to solve the recurrence relation. ■

First note there is the unspoken rule that if a digit is $4,5,6,7,8$ or 9, then there is no restriction on what the next one should be. Let's take a string of n length. Write it of the form

$$k_1 \cdot 10^n + k_2 \cdot 10^{n-1} + k_3 \cdot 10^{n-2} + \cdots$$

so k_i is he $i-$th digit in this string. Now let $n = 6$. Suppose $k_1 \in \{4,5,6,7,8,9\}$. Then k_2 can be anything, so the number of $6-$strings starting with any one of $k_1 \in \{4,5,6,7,8,9\}$ is $6b_{n-1}$. Now suppose the string starts with a 3. Then then $k_2 \in \{4,5\}$. Since these possibilites themselves have no restrictions on the next digits, the total valid $6-$strings that begin with a 3 is given by $2b_{n-2}$. Lastly suppose $k_1 \in \{0,1,2\}$. Then the next has to be a 3 or a 4. If it is a 4, there are no restrictions for k_3 so there are $3b_{n-2}$ possibilites for a string starting with $k_1 \in \{0,1,2\}$ and $k_2 = 4$. But what if $k_2 = 3$? Then $k_3 \in \{4,5\}$. So there are $6b_{n-3}$ possibilites in this case. Since all these are disjoint we have

$$b_n = 6b_{n-1} + 2b_{n-2} + 3b_{n-2} + 6b_{n-3}$$

or in a nicer way:

$$b_n = 6b_{n-1} + 5b_{n-2} + 6b_{n-3}$$

The initial conditions are $b_0 = 1, b_1 = 10$, and $b_2 = 68$.

Exercise 3.6 The sequence c_n is defined by the recursive relation

$$c_n = 4c_{n-1} + (-1)^{n-1}$$

for $n \geq 1$, with initial condition $c_0 = 2$. Find the generating function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and use this to find an explicit formula for c_n. ■

Theorem 3.2.1 We have the following equations:

1.
$$\sum_{n=0}^{\infty} k^n x^n = \frac{1}{1-kx}$$

2.
$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

3.
$$\sum_{n=0}^{\infty} (n+1)k^n x^n = \frac{1}{(1-kx)^2}$$

Proof. Course lecture topic 6. ■

We have

$$\begin{aligned}
f(x) &= c_0 + \sum_{n=1}^{\infty} c_n x^n \\
&= 2 + \sum_{n=1}^{\infty} \left(4c_{n-1} + (-1)^{n-1}\right) x^n \\
&= 2 + \sum_{n=1}^{\infty} 4c_{n-1}x^n + \sum_{n=1}^{\infty} (-1)^{n-1}x^n \\
&= 2 + 4x\sum_{n=1}^{\infty} c_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} (-1)^{n-1}x^{n-1} \\
&= 2 + 4x\sum_{n=0}^{\infty} c_n x^n + x\sum_{n=0}^{\infty} (-1)^n x^n
\end{aligned}$$

and using theorem 3.2.1 we finally get

$$\begin{aligned}
f(x) &= 2 + 4x\sum_{n=0}^{\infty} c_n x^n + x\sum_{n=0}^{\infty} (-1)^n x^n \\
&= 2 + 4xf(x) + \frac{x}{1+x} \\
f(x) - 4xf(x) &= \frac{2(1+x)}{(1+x)} + \frac{x}{1+x} \\
f(x)(1-4x) &= \frac{2+3x}{1+x} \\
f(x) &= \frac{2+3x}{(1+x)(1-4x)}
\end{aligned}$$

Thankfully partial fraction decomposition in this case is quite easy:

$$\begin{aligned}
f(x) &= \frac{A}{1-4x} + \frac{B}{1+x} \\
&= \frac{A(1+x)}{(1+x)(1-4x)} + \frac{B(1-4x)}{(1+x)(1-4x)} \\
&= \frac{A+Ax+B-B4x}{(1+x)(1-4x)} \\
&= \frac{(A+B)+(A-4B)x}{(1+x)(1-4x)}
\end{aligned}$$

and matching the numerators we get the system

$$\begin{aligned}
A+B &= 2 \\
A-4B &= 3
\end{aligned}$$

subtracting the second from the first we get $5B = -1$ so $B = -1/5$. This implies $A = 2 + 1/5 = 10/5 + 1/5 = 11/5$. In total our decomposition shows

$$f(x) = \frac{2+3x}{(1+x)(1-4x)} = \frac{11}{5(1-4x)} - \frac{1}{5(1+x)} = \frac{1}{5}\left(\frac{11}{1-4x} - \frac{1}{1+x}\right)$$

and we can manipulate as so:

$$\begin{aligned} f(x) &= \frac{1}{5}\left(11\sum_{n=0}^{\infty} 4^n x^n - \sum_{n=0}^{\infty}(-1)^n x^n\right) \\ &= \frac{1}{5}\left(\sum_{n=0}^{\infty} 11\cdot 4^n x^n - \sum_{n=0}^{\infty}(-1)^n x^n\right) \\ &= \sum_{n=0}^{\infty} \frac{11\cdot 4^n - (-1)^n}{5} x^n \end{aligned}$$

and comparing this with our original terms we get

$$c_n = \frac{11\cdot 4^n - (-1)^n}{5}$$

Exercise 3.7 Let e_n be the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 = n$$

where x_3 is even.

1. Write down the generating function $f(x) = \sum_{n=0}^{\infty} e_n x^n$, as a rational function in x.
2. Use the generating function to find an explicit formula for e_n.

■

As we have done before, the only rstriction on x_3 is we require it be even, thus it's generating factor is

$$G_3(x) = \frac{1}{1-x^2}$$

so our generating function is

$$f(x) = \frac{1}{(1-x)^2(1-x^2)} = \frac{1}{(1-x)^2(1-x)(1+x)} = \frac{1}{(1-x)^3(1+x)}$$

Now for (2) we will do the partial fraction decomposition like so:

$$f(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1+x}$$

Substituting $x = -1$ using cover up method for D we see $D = 1/8$. Doing the same for C we substitute $x = 1$ and find $C = 1/2$. Now to find out A and B we can't use cover up method, so remember in total we have

$$f(x) = \frac{1}{(1-x)^3(1+x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{1}{2(1-x)^3} + \frac{1}{8(1+x)}$$

and multiplying both sides by $(1-x)^3(1+x)$ we get

$$1 = A(1-x)^2(1+x) + B(1-x)(1+x) + \frac{1}{2}(1+x) + \frac{1}{8}(1-x)^3$$

To find A and B let us substitute $x = 0$ to get

$$1 = A + B + \frac{5}{8} \qquad \Longrightarrow \qquad A + B = \frac{3}{8}$$

We can also substitute $x = 2$ to get

$$1 = 3A - 3B + \frac{3}{2} - \frac{1}{8} = 3A - 3B + \frac{11}{8} \quad \Longrightarrow \quad -\frac{3}{8} = 3A - 3B \quad \Longrightarrow \quad A - B = -\frac{1}{8}$$

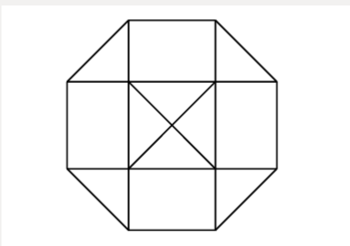
Subtracting the second equation from the first we find $2B = 2/8$ so $B = 1/4$. This implies $A = 1/8$. We can now manipulate $f(x)$ like so:

$$\begin{aligned} f(x) &= \frac{1}{8}\frac{1}{(1-x)} + \frac{1}{4}\frac{1}{(1-x)^2} + \frac{1}{2}\frac{1}{(1-x)^3} + \frac{1}{8}\frac{1}{(1+x)} \\ &= \frac{1}{8}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n + \frac{1}{8}\sum_{n=0}^{\infty}(-1)^n x^n \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{8} + \frac{n+1}{4} + \frac{(n+1)(n+2)}{4} + \frac{(-1)^n}{8}\right)x^n \end{aligned}$$

and by matching we see e_n is given by

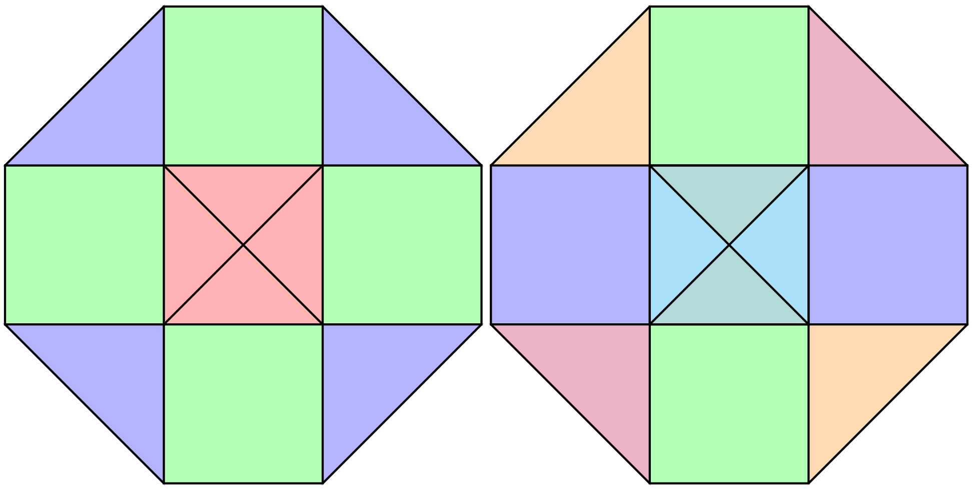
$$\begin{aligned} e_n &= \frac{1}{8} + \frac{n+1}{4} + \frac{(n+1)(n+2)}{4} + \frac{(-1)^n}{8} \\ &= \frac{1+(-1)^n}{8} + \frac{(n+1)+(n+1)(n+2)}{4} \\ &= \frac{1+(-1)^n}{8} + \frac{(n+1)(n+3)}{4} \end{aligned}$$

Exercise 3.8 An artist makes stained glass ornaments by joining 12 pieces of glass together to form a regular octagon, as shown in the figure below. How many inequivelant ornaments can the artist make if there are k different colours of glass available, and two ornaments are considered equivelant if they are related by a rotation or reflection?

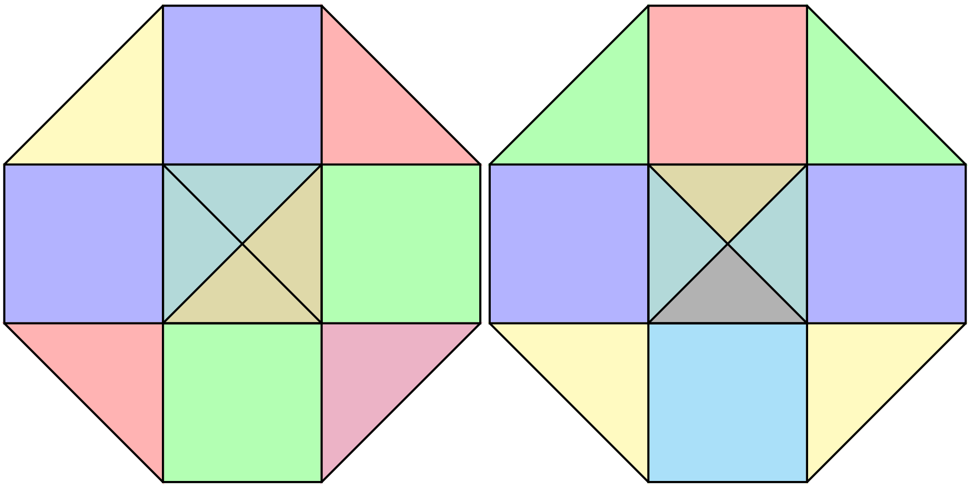


For full credit you must provide justification for the sizes of your fixed point sets. This could be done for example by drawing pictures. ■

Finally some group theory! Here we have $G = D_4$. This has 4 reflections and 4 rotations, in which the only non-trivial rotations are $\pm\pi/2$ and π radians. In terms of reflections we have the diagonal and horizontal-vertical axes. Clearly for the identity we have k^{12} ways to color this ornament. Here is the coloring for the $\pm\pi/2$ rotation:



As you can see for the one that accounts for rotations for $\pm\pi/2$ we have $|\mathscr{C}(f)| = 3$. As for our full 180° rotation we have $|\mathscr{C}(f)| = 6$. Now we show the fixed ornaments by diagonal and vertical reflection. Here the diagonal is from top left to bottom right. We also observe a vertical reflection through the middle of the ornament.



In the left figure we count $|\mathscr{C}(f)| = 7$ and in the right figure we have $|\mathscr{C}(f)| = 8$ as well. The information can be summarised in total as the following:

group element type	number	$\lvert\mathscr{C}(f)\rvert$	total
identity	1	c^{12}	c^{12}
$\pm\frac{\pi}{2}$ rotation	2	c^3	$2c^3$
π rotation	1	c^6	c^6
diagonal reflection	2	c^7	$2c^7$
horizontal/vertical reflection	2	c^8	$2c^8$

By Burnside's Theorem the number of inequivalent ornaments is

$$\frac{1}{|G|}\sum_{f\in G}|C(f)| = \frac{1}{|D_4|}\sum_{f\in D_4}|C(f)| = \frac{c^{12}+2c^8+2c^7+c^6+2c^3}{8}.$$