A CHEBYSHEV TRACE CRITERION FOR THE RIEMANN HYPOTHESIS

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ABSTRACT. We give a self-contained synthesis of an order-three descent of the Riemann xi function and isolate the exact point at which its earlier positivity criteria become a boundedness criterion. The anharmonic group generated by

$$w(s) = \frac{s-1}{s}, \qquad \sigma(s) = 1 - s$$

has invariant

$$q(s) = \frac{(s-2)(s+1)(2s-1)}{2s(s-1)}, \qquad q(s)^2 + \frac{27}{4} = \frac{(s^2-s+1)^3}{s^2(s-1)^2}.$$

The orbit product of ξ descends to an entire function $K(q^2)$ of order $1/2$ and genus zero. Nontrivial critical-line zeros are transported to $(-\infty, -c)$, $c = 27/4$, while $K(x) > 0$ for $x \geq 0$. This yields, without zero ordinates, one local moment sequence whose equivalent forms are an unshifted Hankel tower, a Hausdorff moment sequence, a compact Jacobi realization, a two-index beta lattice (Pascal form), and a completely monotone boundary.

We then make a lossless triangular change of basis. For the normalized zero nodes $\vartheta_j = -c/x_j$, put $y_j = 1 - 2\vartheta_j$ and

$$u_n = \sum_j \vartheta_j T_n(y_j).$$

Every u_n is an integer linear combination of the local moments and therefore a finite residue functional of ξ'/ξ over the single exceptional orbit $\{-1, 1/2, 2\}$. Writing $u_0 = h_0 > 0$, we prove

$$\text{RH} \iff |u_n| \leq h_0 \ \forall n \iff \sup_n |u_n| < \infty \iff \limsup_{n\to\infty} |u_n|^{1/n} \leq 1.$$

The reverse implication is carried by one rigidity fact: escaping poles cannot cancel. Its algebraic core is

$$1 - 2vy_j + v^2 = (1-v)^2 \left(1 - \frac{X(v)}{x_j}\right), \qquad X(v) = \frac{27v}{(1-v)^2},$$

so a trace pole is exactly a Koebe preimage of a zero of K. An off-line zero therefore becomes an uncancellable pole inside the unit disk and forces exponential coefficient growth. The exact exponent is the Green depth of the transported zero.

Finally, a uniform Bernstein–Walsh theorem shows that Chebyshev is exponent-optimal, up to an absolute factor, among all degree-n probes retaining finite three-point locality. At a hypothetical zero $1/2 + \eta + i\gamma$, this gives the intrinsic local e-folding scale $n \asymp \gamma^2/|\eta|$. A scale-dependent Möbius scan improves the conformal exponent to order $|\eta|/\gamma$, and we prove exactly why that improvement costs finite locality. The infinite verification remains infinite; what changes is the form of failure, from a hidden sign defect to an explicit interior singularity.

1. Introduction: what changes, and what does not

The point of an exact reformulation of the Riemann Hypothesis is not to move the same difficulty behind new notation. It is to expose a mechanism that the original coordinates hide. The progression studied here begins with a genus-zero entire function and an RH-equivalent Hankel tower, passes through Hausdorff moments, a compact Jacobi operator, a two-index beta lattice

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(Pascal form), and a one-dimensional boundary sequence, and ends with a qualitatively different coordinate: after changing from powers to the Chebyshev basis, leaving the critical line creates an interior pole.

The logical spine is short.

- **A. Unconditional algebra.** The order-three orbit, the invariant q, the critical value $c = 27/4$, the quotient identity, and the three exceptional residue points are exact rational identities.
- **B. Unconditional descent.** The orbit norm of ξ descends to a genus-zero entire function K; its zeros are the transported nontrivial zeros.
- **C. Equivalent positivity coordinates.** RH is equivalent to one Hankel tower, one Hausdorff sequence, a compact positive Jacobi realization, the full two-index beta lattice (Pascal form), and one completely monotone boundary.
- **D. Lossless trace coordinate.** The Chebyshev trace is a lower-triangular invertible transform of the local moments. No finite-order information is discarded.
- **E. Pole rigidity.** The Koebe factorization sends every off-line transported zero to a genuine, non-cancellable pole in $\mathbb{D}$.
- **F. Equivalent boundedness coordinate.** RH is therefore equivalent to boundedness of one real sequence, or to holomorphy/positive-real behavior of one function in $\mathbb{D}$.
- **G. Unconditional resolution theorem.** Bernstein–Walsh bounds every finite-local polynomial probe; Chebyshev reaches the optimal exponential scale. A Möbius scan improves that scale only by changing the information model.
- **H. Open arithmetic step.** A direct proof of the completed prime–archimedean trace bound would imply RH; that inequality is not proved here.

No statement in A, B, D, E, or G assumes RH. Sections involving a positive spectral measure are explicitly conditional. Every equivalence is proved in both directions.

Theorem 1.1 (Definitive equivalence chain)**.** *Let K, (m_n), (h_n), $D_{r,k}$, (b_k), and (u_n) be defined below. Then the following are equivalent:*

(1) *RH;*
(2) *$Z(K) \subset (-\infty, -c)$, equivalently every zero of K is real (the last paragraph of the proof of Theorem 2.4 excludes the remaining real locus);*
(3) *every Hankel matrix $[m_{i+j}]_{i,j=0}^N$ is positive semidefinite;*
(4) *(h_n) is a Hausdorff moment sequence on $[0,1]$, equivalently $D_{r,k} \geq 0$ for every $r, k \geq 0$;*
(5) *(h_n/h_0) is the moment sequence of a compact positive Jacobi contraction with a cyclic vector;*
(6) *the boundary $b_k = D_{0,k}$ is completely monotone;*
(7) *$|u_n| \leq h_0$ for every $n \geq 0$;*
(8) *$\sup_n |u_n| < \infty$;*
(9) *$\mathcal{U}(v) = \sum_{n\geq 0} u_n v^n$ is holomorphic on $\mathbb{D}$;*
(10) *the centered trace $\mathcal{C}$ defined in Section 5 is Carathéodory on $\mathbb{D}$;*
(11) *its Cayley transform S defined in Section 7 is Schur on $\mathbb{D}$.*

If RH fails, then

$$\limsup_{n\to\infty} \frac{1}{n} \log |u_n| = \max_j g(y_j) > 0,$$

where g is the Green function of $\mathbb{C} \setminus [-1, 1]$.

Inside this fixed family, "collapse" means that the full positivity conditions and the scalar boundedness condition define the same zero set. No generic assertion that bounded Fourier coefficients imply positivity is being made. The exceptional reverse implication is pole rigidity: the admissible meromorphic family has prescribed poles, and an escaping pole cannot cancel.

For orientation, the main symbols are

q	degree-three invariant of the cyclic orbit
$c = 27/4$	nonzero finite critical value of the quotient
K	$\Phi(s) = K(q(s)^2)$
x_j	zeros of K
$\vartheta_j = -c/x_j,\ y_j = 1 - 2\vartheta_j$	compactified zero nodes
h_n	normalized local moments
$D_{r,k},\ b_k$	two-index beta lattice (Pascal form) and its boundary
u_n	Chebyshev trace.

Remark 1.2 (Dependency ledger). All algebraic identities, the lossless trace transform, pole rigidity, the boundedness theorem, the finite-local frontier, the Möbius tradeoff, and the classical function-theoretic reformulations are proved here from the stated hypotheses and standard theorems. No numerical calculation enters any proof. The companion manuscript [10] is cited only for the previously proved beta-lattice positive wedge and related historical context; those statements are not used to prove the present boundedness theorem.

2. The descent is forced

Let

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

so ξ is entire of order one, real on $\mathbb{R}$, satisfies $\xi(1-s) = \xi(s)$, and its zeros are precisely the nontrivial zeros of ζ. Put

$$q(s) = \frac{(s-2)(s+1)(2s-1)}{2s(s-1)} = s - \frac{1}{2} - \frac{1}{s} - \frac{1}{s-1}, \qquad c = \frac{27}{4}. \tag{1}$$

Let

$$w(s) = \frac{s-1}{s}, \qquad \sigma(s) = 1-s.$$

Then $w^3 = \sigma^2 = \mathrm{id}$ and $\sigma w \sigma = w^2$.

Lemma 2.1 (Basic invariance)**.**

$$q \circ w = q, \qquad q \circ \sigma = -q, \qquad \sigma \circ w = w^2 \circ \sigma.$$

Consequently q is the degree-three quotient by $\langle w \rangle$, and q^2 is invariant under the full anharmonic group S_3.

Proof. Direct substitution in (1); the generic fiber of q is $\{s, w(s), w^2(s)\}$. □

Lemma 2.2 (Quotient identity and critical value)**.** *Let $\lambda = s^2 - s + 1$ and $h(\lambda) = \lambda^3/(\lambda-1)^2$. Then*

$$q(s)^2 + c = h(\lambda) = \frac{(s^2-s+1)^3}{s^2(s-1)^2}. \tag{2}$$

Moreover $h'(\lambda) = \lambda^2(\lambda-3)/(\lambda-1)^3$, so $c = 27/4 = h(3)$ is the unique nonzero finite critical value, and

$$4\lambda^3 - 27(\lambda-1)^2 = (\lambda-3)^2(4\lambda-3).$$

Proof. Put $u = s(s-1)$ and $m = s - 1/2$. Then $u = m^2 - 1/4$ and $q = m(u-2)/u$. Hence

$$u^2(q^2+c) = (u+1/4)(u-2)^2 + \frac{27}{4}u^2 = (u+1)^3.$$

Since $u+1 = \lambda$ and $u = \lambda - 1$, the identity and derivative follow. □

Proposition 2.3 (Distinguished fibers)**.**

$$q^{-1}(0) = \{-1, 1/2, 2\}, \qquad q^{-1}(\infty) = \{0, 1, \infty\}.$$

Each is a single w-orbit. The fixed points of w are $e^{\pm i\pi/3}$ and map under q^2 to $-c$. Thus q^2 has the three classical branch values $-c, 0, \infty$.

Proof. Read the zero and polar fibers from (1); the fixed-point equation is $s^2 - s + 1 = 0$, and (2) then gives $q^2 = -c$. □

The modular interpretation is automatic: in the classical modular-λ coordinate,

$$j_\lambda(s) = 256\frac{(s^2 - s + 1)^3}{s^2(s-1)^2} = 256(q(s)^2 + c), \qquad 256c = 1728.$$

We use this only as geometric context.

Define

$$\Phi(s) = \xi(s)\xi(w(s))\xi(w^2(s)).$$

Theorem 2.4 (Anharmonic descent)**.** *The function Φ is invariant under the anharmonic group and there is an entire function K with real Taylor coefficients such that*

$$\Phi(s) = K(q(s)^2). \tag{3}$$

The function K has order exactly $1/2$ and genus zero. Hence

$$K(x) = K(0)\prod_j \left(1 - \frac{x}{x_j}\right), \qquad \sum_j \frac{1}{|x_j|} < \infty, \tag{4}$$

with

$$\frac{K'(x)}{K(x)} = \sum_j \frac{1}{x - x_j}. \tag{5}$$

Every zero x_j equals $q(\rho)^2$ for at least one nontrivial zero ρ, and its multiplicity is the total ξ-zero multiplicity carried by the corresponding cyclic orbit. Every zero of K arises this way. Moreover

$$K(x) > 0 \qquad (x \geq 0), \tag{6}$$

and

$$\mathrm{RH} \iff Z(K) \subset (-\infty, -c) \iff Z(K) \subset \mathbb{R}. \tag{7}$$

On the critical line,

$$q(1/2 + it) = iB(t), \qquad B(t) = t\frac{4t^2 + 9}{4t^2 + 1}. \tag{8}$$

Proof. Cyclic invariance is immediate. The identities $w(1-s) = 1-w^2(s)$ and $w^2(1-s) = 1-w(s)$, together with $\xi(1-z) = \xi(z)$, give invariance under σ. The finite-group quotient theorem for compact Riemann surfaces [4] gives the holomorphic factorization through q^2; finite fibers avoid the polar orbit, so K is entire. Conjugation gives real Taylor coefficients.

The branch fibers carry no nontrivial zeta zero, so every zero-bearing fiber is unramified and multiplicities descend by summing the zero multiplicities over the cyclic orbit. As $s \to \infty$, $q(s) = s - 1/2 + O(s^{-1})$, while the other two orbit points tend to 1 and 0. An inverse branch of $q(s)^2 = x$ therefore has $|s| \asymp |x|^{1/2}$; since ξ has order one, K has order at most $1/2$. Conversely Riemann–von Mangoldt zero counting [13] and the finite fiber bound give order at least $1/2$. Thus the order is exactly $1/2$, and Hadamard factorization has genus zero.

For $x \geq 0$, set $Q^2 = x$. Clearing denominators in $q(z) = Q$ gives a real cubic whose discriminant is $(Q^2 + c)^2 > 0$, so its fiber consists of three distinct real points. The completed xi function is positive on the real line, hence $K(x) > 0$. Finally, direct calculation gives

$$\Re q(\sigma + it) = \frac{(2\sigma - 1)(|s|^2 - 1)(|1 - s|^2 - 1)}{2|s|^2|1 - s|^2},$$

while $\Im q(s) = t(1 + |s|^{-2} + |1 - s|^{-2}) \neq 0$ for a nontrivial zero. Thus $q(\rho)^2$ is real exactly on the critical line. The positive axis is excluded by (6), and the nontrivial zero height forces $B(\gamma)^2 > c$, giving the ray $(-\infty, -c)$. □

3. From zero location to the positivity tower

Throughout this section (x_j) denotes the multiset of zeros of K, with multiplicity. We use the index j when only the descended zero is relevant, and ρ when retaining a source zero of ξ is convenient; the latter notation is essential when height or horizontal displacement matters.

Definition 3.1 (Moments and normalized nodes)**.** Define

$$m_n = \frac{(-1)^n}{n!} \frac{d^n}{dx^n} \frac{K'(x)}{K(x)}\bigg|_{x=0}, \qquad h_n = c^{n+1} m_n, \tag{9}$$

and

$$\vartheta_j = -\frac{c}{x_j}, \qquad y_j = 1 - 2\vartheta_j. \tag{10}$$

Then, unconditionally,

$$h_n = \sum_j \vartheta_j^{n+1}, \qquad \sum_j |\vartheta_j| < \infty. \tag{11}$$

Lemma 3.2 (Node reality)**.** *If $x_j = q(\rho)^2$ comes from $\rho = \sigma + it$, then*

$$\vartheta_j \in \mathbb{R} \iff \sigma = 1/2.$$

On the critical line $0 < \vartheta_j < 1$, hence $y_j \in (-1, 1)$; off the line both are nonreal.

Proof. The real and imaginary parts of q are

$$\Im q(s) = t\left(1 + \frac{1}{|s|^2} + \frac{1}{|1 - s|^2}\right), \qquad \Re q(s) = \frac{(2\sigma - 1)(|s|^2 - 1)(|1 - s|^2 - 1)}{2|s|^2|1 - s|^2}.$$

For nontrivial zeros $0 < \sigma < 1$ and $|t| > 1$, so $q(\rho)$ is never real and is purely imaginary exactly on $\sigma = 1/2$. On the line (8) gives $\vartheta = c/B(\gamma)^2 \in (0, 1)$. □

Theorem 3.3 (Three-point locality)**.** *Let $\ell = \xi'/\xi$. For every $n \geq 0$,*

$$m_n = \frac{(-1)^n}{2} \sum_{a \in \{-1, 1/2, 2\}} \operatorname*{Res}_{s=a} \frac{\ell(s)}{q(s)^{2n+2}}. \tag{12}$$

Thus every finite polynomial combination of the moments is computable from finitely many derivatives of ℓ at $-1, 1/2, 2$; no zero ordinate enters its definition.

Proof. Let C_x be a sufficiently small positively oriented circle about $x = 0$, containing no zero of K. By Cauchy's coefficient formula and the definition of m_n,

$$(-1)^n m_n = \frac{1}{2\pi i} \int_{C_x} \frac{K'(x)}{K(x)} \frac{dx}{x^{n+1}}.$$

Write $\Psi(q) = K(q^2)$. Since

$$\frac{\Psi'(q)}{\Psi(q)} = 2q \frac{K'(q^2)}{K(q^2)},$$

the factors $2q$ from $dx = 2q\,dq$ and from Ψ'/Ψ cancel. A single positively oriented q-circle maps twice around C_x under $x = q^2$, hence

$$(-1)^n m_n = \frac{1}{2}\frac{1}{2\pi i}\int_{C_q} \frac{\Psi'(q)}{\Psi(q)}\frac{dq}{q^{2n+2}}. \tag{13}$$

The three zeros $-1, 1/2, 2$ of q are simple. A small q-circle therefore has three local inverse lifts C_a, one about each $a \in \{-1, 1/2, 2\}$, each mapping once and with positive orientation to C_q. Since $\Phi(s) = \Psi(q(s))$,

$$\frac{\Psi'(q)}{\Psi(q)}\,dq = \frac{\Phi'(s)}{\Phi(s)}\,ds,$$

and averaging the three identical lifts gives

$$\frac{1}{2\pi i}\int_{C_q} \frac{\Psi'(q)}{\Psi(q)}\frac{dq}{q^{2n+2}} = \frac{1}{3}\sum_a \operatorname*{Res}_{s=a} \frac{\Phi'(s)/\Phi(s)}{q(s)^{2n+2}}.$$

Finally,

$$\frac{\Phi'(s)}{\Phi(s)} = \sum_{r=0}^{2} \ell(w^r(s))(w^r)'(s).$$

Because w cyclically permutes $-1, 2, 1/2$ and leaves q invariant, the total residue sum over the three points is the same for each of the three orbit terms. Thus

$$\sum_a \operatorname*{Res}_{s=a} \frac{\Phi'(s)/\Phi(s)}{q(s)^{2n+2}} = 3\sum_a \operatorname*{Res}_{s=a} \frac{\ell(s)}{q(s)^{2n+2}}.$$

The factor 3 cancels the preceding $1/3$, while the factor $1/2$ in (13) remains. This proves (12). $\square$

Corollary 3.4 (Positive normalization)**.** *The constant h_0 is unconditionally positive. In fact*

$$h_0 = \frac{4}{3}(\ell(2) + \ell'(2)) + \frac{1}{24}\ell'(1/2) > 0. \tag{14}$$

Proof. Expanding the three residues in (12) gives (14). For completeness, use the classical positive-kernel representation of the Riemann Ξ-function (see, for example, [13]) to write

$$M(u) := \xi\left(\frac{1}{2} + u\right) = \int_{\mathbb{R}} e^{ut}\,d\mu(t),$$

where μ is a nonzero positive even measure and is not supported at a single point. Therefore

$$(\log M)''(u) = \frac{\int t^2 e^{ut}\,d\mu}{\int e^{ut}\,d\mu} - \left(\frac{\int t e^{ut}\,d\mu}{\int e^{ut}\,d\mu}\right)^2 > 0.$$

Thus $\log M$ is strictly convex and even. Consequently $\ell(1/2) = 0$, $\ell'(1/2) > 0$, $\ell'(2) > 0$, and $\ell(2) > 0$. Every term on the right of (14) is therefore positive. $\square$

Remark 3.5 (Definition side versus zero side)*.* The moments are defined by derivatives of K'/K at $x = 0$ and then by the local residue formula. The zero sum (11) follows from genus zero; it is not supplied as input. Zero-side identities are used to prove and interpret the criterion, not to define it.

Theorem 3.6 (One Hankel tower)**.** *Let $H_N = [m_{i+j}]_{i,j=0}^N$. Then*

$$\text{RH} \iff H_N \succeq 0 \qquad (N \geq 0).$$

Proof. Under RH, with $a_j = -1/x_j > 0$, $m_n = \sum_j a_j^{n+1}$, so each quadratic form is $\sum_j a_j p(a_j)^2 \geq 0$. Conversely, positivity of every Hankel matrix gives by the Hamburger moment theorem [1] a positive real measure μ with moments (m_n). Put

$$f(z) = \frac{K'(-z)}{K(-z)} = \sum_{n\geq 0} m_n z^n, \qquad r_0 = \min_j |x_j| > 0.$$

For every $0 < r < r_0$, Cauchy's estimate gives

$$|m_n| \leq M_r r^{-n}, \qquad M_r = \max_{|z|=r} |f(z)|. \tag{15}$$

This forces every representing measure to have compact support. Indeed, if $\mu(\{|t| \geq A\}) = \varepsilon > 0$ for some $A > 1/r_0$, choose r with $1/A < r < r_0$. Then

$$m_{2n} = \int t^{2n}\, d\mu(t) \geq \varepsilon A^{2n},$$

contradicting (15) as $n \to \infty$. Hence $\operatorname{supp}\mu \subset [-1/r_0, 1/r_0]$; in particular the moment problem is determinate. On RH, $r_0 = B(\gamma_1)^2$, so this becomes the sharp support bound $|t| \leq B(\gamma_1)^{-2}$. Its Cauchy transform

$$\mathfrak{s}_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{1 - tz}$$

agrees with $f(z)$ near the origin and is holomorphic off the real reciprocal support. By analytic continuation a nonreal zero of K would create a nonremovable nonreal pole of f, impossible for $\mathfrak{s}_\mu$. Hence every zero of K is real, and (7) gives RH. $\square$

For $r, k \geq 0$ define

$$D_{r,k} = (-1)^k \Delta^k h_r = \sum_{j=0}^{k} (-1)^j \binom{k}{j} h_{r+j}, \qquad b_k = D_{0,k}. \tag{16}$$

Theorem 3.7 (Hausdorff, beta, and boundary forms)**.** *The following are equivalent:*

(1) *RH;*
(2) (h_n) *is a Hausdorff moment sequence on* $[0,1]$*;*
(3) $D_{r,k} \geq 0$ *for all* $r, k \geq 0$*;*
(4) (b_k) *is completely monotone.*

Unconditionally,

$$D_{r,k} = \sum_j \vartheta_j^{r+1} (1 - \vartheta_j)^k, \tag{17}$$

and

$$D_{r,k} = D_{r+1,k} + D_{r,k+1}, \qquad D_{r,k} = (-1)^r \Delta^r b_k. \tag{18}$$

On RH,

$$h_n = \int_0^1 t^n\, d\nu(t), \qquad \nu = \sum_j \vartheta_j\, \delta_{\vartheta_j}. \tag{19}$$

Proof. Equation (17) is the binomial transform of (11). Under RH, $0 < \vartheta_j < 1$, which gives (19) and complete monotonicity. Conversely, the Hausdorff theorem [1] gives a unique positive measure ν on $[0,1]$ with moments (h_n). Its generating function

$$\mathfrak{g}_\nu(z) = \int_0^1 \frac{d\nu(t)}{1 - tz} = \sum_{n\geq 0} h_n z^n$$

is holomorphic on $\mathbb{C} \setminus [1, \infty)$. From (9) the same Taylor germ is $cK'(-cz)/K(-cz)$. The identity theorem therefore continues the two functions throughout their common domain, so every pole of the logarithmic derivative must lie on the positive real cut. Equivalently every zero of K lies on $(-\infty, -c]$. The strict endpoint $-c$ contains no nontrivial zero, hence (7) gives RH. Pascal's identity yields (18) and the boundary equivalence. □

Proposition 3.8 (Boundary generating function)**.** *For $|z|$ sufficiently small,*

$$\mathcal{B}(z) := \sum_{k \geq 0} b_k z^k = \frac{c}{1-z} \frac{K'\left(\frac{cz}{1-z}\right)}{K\left(\frac{cz}{1-z}\right)}. \tag{20}$$

With $x = cz/(1-z)$, this is $\mathcal{B}(z) = (x+c)K'(x)/K(x)$. The identity extends wherever both sides analytically continue.

Theorem 3.9 (Compact Jacobi realization)**.** *RH is equivalent to the existence of a compact positive Jacobi contraction J on $\ell^2(\mathbb{N}_0)$ with cyclic vector e_0 such that*

$$\frac{h_n}{h_0} = \langle e_0, J^n e_0 \rangle.$$

Under RH its nonzero spectrum is precisely $\{\vartheta_j\}$, with multiplicity. The standard Jacobi coefficients for the m-moment measure are given by the Hankel determinant ratios; the compact normalized operator above is the corresponding c-scaled realization because $h_n/h_0 = c^n m_n/m_0$.

Proof. Normalize the Hausdorff measure (19) by h_0 and represent multiplication by t in its orthogonal-polynomial basis. Since its atoms tend to 0, the operator is compact. Conversely, the spectral measure of a positive contraction is supported on $[0, 1]$, so its moments are Hausdorff and Theorem 3.7 gives RH. □

The companion analysis [10] proves the unconditional contextual statement that for every fixed $\varepsilon > 0$, $D_{r,k} > 0$ for all sufficiently large k whenever $r + 1 \leq k^{1/2-\varepsilon}$. This theorem is not used below; it records that the positivity coordinates already identify a genuine region in which failure cannot occur.

4. The lossless Chebyshev coordinate

Let T_n denote the Chebyshev polynomial of the first kind,

$$T_n(\cos \phi) = \cos(n\phi).$$

Define

$$R_n(t) = tT_n(1 - 2t). \tag{21}$$

Because T_n has integer coefficients and satisfies

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

the polynomial R_n has integer coefficients, vanishes at $t = 0$, and obeys

$$R_{n+1}(t) = 2(1 - 2t)R_n(t) - R_{n-1}(t). \tag{22}$$

Write

$$R_n(t) = \sum_{k=0}^{n} a_{n,k} t^{k+1}, \qquad a_{n,k} \in \mathbb{Z}. \tag{23}$$

Definition 4.1 (Circle trace)**.** For $n \geq 0$, define

$$u_n = \sum_{k=0}^{n} a_{n,k} h_k. \tag{24}$$

Proposition 4.2 (Lossless finite-level change of basis)**.** *For every $N \geq 0$, the map*

$$(h_0, \dots, h_N) \longmapsto (u_0, \dots, u_N)$$

is lower triangular with nonzero diagonal. More precisely, $a_{0,0} = 1$ and, for $n \geq 1$,

$$a_{n,n} = (-1)^n 2^{2n-1}.$$

Hence the transform is invertible over $\mathbb{Q}$ at every finite level. The trace changes the geometry of the criterion; it does not discard finite-order moment information.

Proof. For $n \geq 1$, the leading coefficient of $T_n(x)$ is 2^{n-1}. Substituting $x = 1 - 2t$ and multiplying by t gives the stated coefficient of t^{n+1}. □

Thus u_n is defined entirely from local data. The zero-side meaning is immediate.

Proposition 4.3 (Zero-mode expansion)**.** *For every $n \geq 0$,*

$$\boxed{u_n = \sum_\rho \vartheta_\rho T_n(y_\rho), \qquad y_\rho = 1 - 2\vartheta_\rho.} \tag{25}$$

The series is absolutely convergent.

Proof. Insert (11) into (24) and use (23). Absolute convergence follows from $\sum |\vartheta_\rho| < \infty$ and the finiteness of R_n. □

The first values are

$$u_0 = h_0, \qquad u_1 = h_0 - 2h_1, \qquad u_2 = h_0 - 8h_1 + 8h_2.$$

For later geometry it is convenient to package each node in one complex angle. Define, with the branch tending to 0 as $\vartheta \to 0$,

$$\alpha_j = 2 \arcsin \sqrt{\vartheta_j}. \tag{26}$$

Then, exactly,

$$\vartheta_j = \sin^2(\alpha_j/2), \qquad y_j = \cos\alpha_j, \qquad T_n(y_j) = \cos(n\alpha_j), \tag{27}$$

and Lemma 3.2 gives

$$\mathrm{RH} \iff \alpha_j \in (0, \pi) \quad \text{for every } j.$$

The trace has a direct residue formula that is more compact than expanding these integer combinations.

Proposition 4.4 (Direct local trace formula)**.** *Put*

$$W(s) = \frac{c}{q(s)^2}.$$

Then

$$\boxed{u_n = \frac{1}{2} \sum_{a \in \{-1, 1/2, 2\}} \operatorname*{Res}_{s=a} \ell(s) W(s) T_n(1 + 2W(s)).} \tag{28}$$

Using the functional equation, this folds to

$$u_n = \left(\operatorname*{Res}_{s=2} + \frac{1}{2} \operatorname*{Res}_{s=1/2} \right) \ell(s) W(s) T_n(1 + 2W(s)). \tag{29}$$

Proof. From (9),

$$h_k = \frac{(-1)^k}{2} \sum_a \operatorname*{Res}_{s=a} \ell(s) W(s)^{k+1}.$$

Substitute this into (24). Since

$$\sum_{k=0}^{n} a_{n,k}(-1)^k W^{k+1} = -R_n(-W) = W T_n(1+2W),$$

we obtain (28). The fold follows from $\ell(1-s) = -\ell(s)$ and $W(1-s) = W(s)$. □

Under RH the trace becomes a genuine Fourier coefficient.

Theorem 4.5 (Circle measure). *Assume RH and write*

$$\phi_\gamma = \arccos(1 - 2\vartheta_\gamma) \in (0, \pi).$$

Then

$$u_n = \int_0^\pi \cos(n\phi)\, d\mu(\phi), \qquad \mu = \sum_{\gamma>0} \operatorname{mult}(\gamma) \vartheta_\gamma \delta_{\phi_\gamma} \geq 0, \tag{30}$$

and $\mu(0, \pi) = h_0$. *Consequently*

$$|u_n| \leq h_0 \qquad (n \geq 0), \tag{31}$$

all Toeplitz matrices

$$[u_{|j-k|}]_{j,k=0}^N \tag{32}$$

are positive semidefinite, and (u_n) *is Bohr almost periodic.*

5. Koebe uniformization and the generating function

The trace is most transparent after a conformal change of variable. Define

$$X(v) = \frac{4cv}{(1-v)^2} = 27 \frac{v}{(1-v)^2}. \tag{33}$$

Up to the scale $4c$, this is the Koebe function $v/(1-v)^2$; it maps $\mathbb{D}$ conformally onto

$$\mathbb{C} \setminus (-\infty, -c]. \tag{34}$$

The endpoint $-c$ is the cubic branch value.

The same map has a useful Joukowski form. If $x = -c/\vartheta$ and $x = X(v)$, then

$$1 - 2\vartheta = \frac{1}{2}\left(v + v^{-1}\right). \tag{35}$$

Thus the unit circle $|v| = 1$ is carried to $[-1, 1]$ by the classical Joukowski map.

Proposition 5.1 (Trace generating identity). *Let*

$$F(x) = x \frac{K'(x)}{K(x)}, \qquad \mathcal{U}(v) = \sum_{n \geq 0} u_n v^n.$$

Near $v = 0$,

$$\boxed{\mathcal{U}(v) = \frac{h_0}{2} + \frac{1-v^2}{8v} F(X(v)).} \tag{36}$$

Equivalently, the centered trace

$$\mathcal{C}(v) := 2\mathcal{U}(v) - h_0 = h_0 + 2 \sum_{n \geq 1} u_n v^n \tag{37}$$

satisfies the unconditional master identity

$$\boxed{\begin{aligned}\mathcal{C}(v) &= \sum_\rho \vartheta_\rho \frac{1-v^2}{1-2vy_\rho+v^2}\\ &= \frac{1-v^2}{4v}F(X(v)).\end{aligned}} \tag{38}$$

Both apparent singularities at $v = 0$ are removable. The two lines of (38) are the same object in zero coordinates and descended-function coordinates, respectively. The algebraic core of the uniformization is the one-node factorization

$$\boxed{1-2vy_\rho+v^2 = (1-v)^2\left(1-\frac{X(v)}{x_\rho}\right), \qquad x_\rho = -\frac{c}{\vartheta_\rho}.} \tag{39}$$

Thus a trace pole is exactly a Koebe preimage of a zero of K.

Proof. From (25) and the Chebyshev generating function,

$$\mathcal{U}(v) = \sum_\rho \vartheta_\rho \frac{1-vy_\rho}{1-2vy_\rho+v^2}.$$

For one node ϑ and $x = -c/\vartheta$, a direct calculation gives

$$\vartheta\frac{1-v(1-2\vartheta)}{1-2v(1-2\vartheta)+v^2} = \frac{\vartheta}{2} + \frac{1-v^2}{8v}\frac{X(v)}{X(v)-x}.$$

Summing and using $h_0 = \sum \vartheta_\rho$ and $F(X) = \sum X/(X-x_j)$ proves the result. ☐

There is a classical modular normalization behind the same map. Put

$$j_\lambda(s) = 256\frac{(s^2-s+1)^3}{s^2(s-1)^2}.$$

Then (2) becomes

$$j_\lambda(s) = 256(q(s)^2+c). \tag{40}$$

This is the usual modular-λ expression for the classical j-invariant, in the normalization $j = 1728J$; see [3, §23.19]. Under (33),

$$256(X(v)+c) = 1728\left(\frac{1+v}{1-v}\right)^2. \tag{41}$$

Thus the unit disk first maps to the right half-plane by a Cayley transform and then, after squaring, to the complement of the negative j-ray. This modular interpretation is context, not a novelty claim.

6. Boundedness, poles, and Green escape

For $y \in \mathbb{C} \setminus [-1,1]$, let

$$\zeta_*(y) = y + \sqrt{y^2-1} \tag{42}$$

be the branch with $|\zeta_*(y)| > 1$, and put

$$g(y) = \log|\zeta_*(y)|. \tag{43}$$

This is the Green function of $\mathbb{C} \setminus [-1,1]$ with pole at infinity; on $[-1,1]$ we set $g = 0$.

Lemma 6.1 (Escaping nodes are sparse)**.** *Along the nontrivial zeros, $\vartheta_\rho \to 0$, hence $y_\rho \to 1$ and $g(y_\rho) \to 0$. For each $\delta > 0$, only finitely many zeros satisfy $g(y_\rho) \geq \delta$.*

Proof. Since $q(\rho) = \rho - 1/2 + O(1/\rho)$ at large height, $|\vartheta_\rho| = c/|q(\rho)|^2 \to 0$. $\square$

Lemma 6.2 (Pole rigidity)**.** *If $y_\rho \notin [-1,1]$, then the term of $\mathcal{U}$ attached to y_ρ has an interior pole at*

$$v_\rho = \zeta_*(y_\rho)^{-1}. \tag{44}$$

After equal node values are merged, this pole is simple and has nonzero residue. Distinct node values give distinct poles. In particular, escaping poles cannot cancel.

Proof. For the Chebyshev denominator,

$$1 - 2vy + v^2 = (1 - v\zeta_*(y))(1 - v/\zeta_*(y)).$$

At $v = \zeta_*(y)^{-1}$ the numerator is

$$1 - \frac{y}{\zeta_*(y)} = \frac{1}{2}\left(1 - \zeta_*(y)^{-2}\right) \neq 0.$$

The node value determines $\vartheta = (1-y)/2$. If a fiber contains m_y zeros, its merged weight is $m_y\vartheta$ with $m_y \in \mathbb{N}$ and $\vartheta \neq 0$, so the residue cannot vanish. Since the Joukowski inverse is injective after the exterior branch is fixed, distinct node values yield distinct interior poles. $\square$

Theorem 6.3 (Radius and boundedness theorem)**.** *The generating function $\mathcal{U}(v)$ extends meromorphically to $\mathbb{D}$, and its only poles there are the genuine simple poles in Lemma 6.2.*

The following are equivalent:

1. *RH;*
2. $|u_n| \le h_0$ *for every* n*;*
3. $\sup_n |u_n| < \infty$*;*
4. $\limsup_{n\to\infty} |u_n|^{1/n} \le 1$*;*
5. $\mathcal{U}$ *is holomorphic on* $\mathbb{D}$*.*

Moreover,

$$\boxed{\limsup_{n\to\infty} \frac{1}{n} \log |u_n| = \sup_\rho g(y_\rho).} \tag{45}$$

The right side is zero exactly on RH and, if positive, its supremum is attained.

Proof. For small $|v|$,

$$\mathcal{U}(v) = \sum_\rho \vartheta_\rho \frac{1 - vy_\rho}{1 - 2vy_\rho + v^2}. \tag{46}$$

Fix $r < 1$. By Lemma 6.1, all but finitely many nodes have $|\zeta_*(y_\rho)|$ arbitrarily close to 1. On $|v| \le r$, the tail in (46) is then uniformly dominated by a constant times $\sum |\vartheta_\rho|$, so it is holomorphic. The remaining finitely many terms are rational. This proves meromorphic continuation and identifies the poles.

RH implies the sharp bound by Theorem 4.5. The implications (2)⇒(3)⇒(4) are immediate. If RH fails, an interior pole makes the radius of convergence < 1, contradicting (4); thus (4)⇒(1). The radius is

$$R = \min\left(1, \exp\left[-\sup_\rho g(y_\rho)\right]\right).$$

Cauchy–Hadamard gives (45). On RH the rate is zero rather than negative because Theorem 8.1 below gives a positive limiting mean square. $\square$

Remark 6.4 (Conformal meaning). Equation (35) says precisely that an escaping zero selects the interior preimage v_ρ of its node under the Joukowski map. Equivalently,

$$g(y_\rho) = -\log |v_\rho|.$$

Thus the exponential growth rate is the conformal depth with which the nearest transported zero penetrates the unit disk.

At large height the Green escape measures horizontal displacement explicitly.

Proposition 6.5 (High-zero escape scale)**.** *Let*

$$\rho = \frac{1}{2} + \eta + i\gamma, \qquad |\eta| < \frac{1}{2},$$

with $\gamma \to \infty$*. Then*

$$g(y_\rho) = \frac{\sqrt{27}\,|\eta|}{\gamma^2}\left(1 + O(\gamma^{-2})\right). \tag{47}$$

Proof. With $z = \gamma - i\eta$, the complex-shift identity gives $\vartheta = c/B(z)^2 = cz^{-2}(1 + O(z^{-2}))$. Near the endpoint $y = 1$, the Joukowski inverse satisfies $\log |\zeta_*(1 - 2\vartheta)| = 2|\operatorname{Im}\sqrt{\vartheta}| + O(|\vartheta|^{3/2})$. Since $2\sqrt{c} = \sqrt{27}$, (47) follows. $\square$

Theorem 6.6 (Uniform finite-local amplification frontier)**.** *Let p be a polynomial of degree at most n with*

$$\|p\|_{[-1,1]} \le 1,$$

and let $y \in \mathbb{C} \setminus [-1, 1]$. Put

$$\zeta_*(y) = y + \sqrt{y^2 - 1}, \qquad |\zeta_*(y)| > 1, \qquad g(y) = \log |\zeta_*(y)|.$$

Then the Bernstein–Walsh inequality gives the finite, uniform bound

$$\boxed{|p(y)| \le e^{ng(y)}.} \tag{48}$$

For the Chebyshev polynomial T_n one has the exact two-sided estimate

$$\boxed{\sinh\big(ng(y)\big) \le |T_n(y)| \le \cosh\big(ng(y)\big).} \tag{49}$$

Consequently, whenever $ng(y) \ge 1$,

$$|T_n(y)| \ge \frac{1 - e^{-2}}{2}\, e^{ng(y)}. \tag{50}$$

Thus Chebyshev is uniformly optimal in exponential scale, up to an absolute multiplicative constant, among all degree-n probes with the same unit bound on $[-1, 1]$.

If

$$U_n[p] = \sum_\rho \vartheta_\rho p(y_\rho), \qquad y_\rho = 1 - 2\vartheta_\rho,$$

then $U_n[p]$ is a finite linear combination of $h_0, \ldots, h_n$. Hence, for a hypothetical zero

$$\rho = \frac{1}{2} + \eta + i\gamma$$

at large height, every normalized finite-local degree-n polynomial probe has single-node amplification at most

$$\exp\left\{n\frac{\sqrt{27}|\eta|}{\gamma^2}\left(1 + O(\gamma^{-2})\right)\right\},$$

while the Chebyshev trace attains this exponential scale uniformly once the exponent is bounded away from zero. In particular, the intrinsic e-folding degree scale for normalized finite-local polynomial amplification is

$$\boxed{n_{\exp}(\rho) \asymp \frac{1}{g(y_\rho)} \asymp \frac{\gamma^2}{|\eta|}.} \tag{51}$$

This is a statement about the exponential rate of an individual normalized probe. It is not the first index at which the full trace must violate $|u_n| \le h_0$.

Proof. Since $y = 1 - 2\vartheta$, the polynomial $\vartheta p(1-2\vartheta)$ has degree at most $n+1$ and vanishes at $\vartheta = 0$. It is therefore a linear combination of $\vartheta, \dots, \vartheta^{n+1}$, and summation over the nodes expresses $U_n[p]$ as a finite linear combination of $h_0, \dots, h_n$.

The Bernstein–Walsh inequality for the compact set $[-1,1]$ is exactly (48); see [9]. For Chebyshev,

$$T_n(y) = \frac{1}{2}\left(\zeta_*(y)^n + \zeta_*(y)^{-n}\right).$$

Writing $r = |\zeta_*(y)| = e^{g(y)}$ and applying the direct and reverse triangle inequalities gives

$$\frac{1}{2}(r^n - r^{-n}) \le |T_n(y)| \le \frac{1}{2}(r^n + r^{-n}),$$

which is (49). If $ng(y) \ge 1$, the lower bound is

$$\sinh(ng) = \frac{1}{2}e^{ng}(1 - e^{-2ng}) \ge \frac{1-e^{-2}}{2}e^{ng},$$

giving (50). Proposition 6.5 then yields (51). $\square$

Remark 6.7 (Exponent versus first visible crossing). The scale $g(y_\rho)^{-1}$ in (51) is the e-folding scale of the exterior polynomial mode, not a universal first-crossing time for the total sequence. A merged escaping node of weight $a_\rho = m_{y_\rho}\vartheta_\rho \neq 0$ contributes $a_\rho T_n(y_\rho)$. Once $ng(y_\rho) \ge 1$, (50) shows that this single contribution reaches a fixed size only after a further residue penalty of order

$$ng(y_\rho) \gtrsim \log \frac{1}{|a_\rho|}.$$

For a simple high zero, $|\vartheta_\rho| \asymp \gamma^{-2}$, so the corresponding single-mode fixed-size onset is of order

$$\frac{\gamma^2}{|\eta|} \log \gamma,$$

up to constants and independently of whatever cancellation or reinforcement is supplied by the remaining modes. This distinction explains why the exact asymptotic growth rate can be small while a numerical trace exhibits a long bounded-looking transient.

Remark 6.8 (What the frontier does and does not say). The theorem is uniform in both the degree and the escaping point; no fixed-y limiting argument is used. The only information restriction is finite three-point locality: the probe is a polynomial in the invariant node $\vartheta = -c/q^2$, so its value is determined by finitely many moments h_k and hence by finitely many derivatives of ξ at $-1, 1/2, 2$.

The theorem does not rule out a construction that changes this information model. In particular, an odd q-channel would contain orientation information discarded by the reflection-invariant descent $q \mapsto q^2$ and is not encoded by the present moment tower. A fixed reweighting of the nodes cannot improve the exponential rate: replacing ϑ_ρ by a weight independent of n changes pole residues but not pole locations, so the Green exponent is unchanged whenever the new weight does not vanish at the escaping node.

Proposition 6.9 (Scale-dependent Möbius scanning family)**.** *Let an escaping pole be written*

$$v_\rho = e^{-g_\rho + i\phi_\rho}, \qquad g_\rho > 0,$$

and for $0 < a < 1$ set

$$\psi_a(w) = \frac{a+w}{1+aw}, \qquad \mathcal{C}_a(w) = \mathcal{C}(\psi_a(w)).$$

The pole of $\mathcal{C}_a$ is

$$w_\rho = \psi_a^{-1}(v_\rho) = \frac{v_\rho - a}{1 - av_\rho}.$$

The moved Green depth has the exact form

$$\boxed{-\log|w_\rho| = \operatorname{artanh}\left(\frac{(1-a^2)\sinh g_\rho}{(1+a^2)\cosh g_\rho - 2a\cos\phi_\rho}\right).} \tag{52}$$

For fixed a and ϕ, its derivative at $g = 0$ is

$$\frac{\partial}{\partial g}\big[-\log|w|\big]\bigg|_{g=0} = P_a(\phi) := \frac{1-a^2}{1+a^2-2a\cos\phi}, \tag{53}$$

the Poisson kernel. For $0 < |\phi| < \pi/2$, the first-order gain $P_a(\phi)$ is maximized over real $0 < a < 1$ at

$$a_*(\phi) = \frac{1-|\sin\phi|}{\cos\phi}, \qquad P_{a_*}(\phi) = \frac{1}{|\sin\phi|}. \tag{54}$$

For a high zero near the critical line,

$$|\phi_\rho| = \frac{\sqrt{27}}{\gamma} + O(\gamma^{-3}), \qquad g_\rho = \frac{\sqrt{27}|\eta|}{\gamma^2}\Big(1 + O(\gamma^{-2})\Big),$$

so the optimally centered real Möbius lens has

$$\boxed{-\log|w_\rho| \sim \frac{g_\rho}{|\sin\phi_\rho|} \sim \frac{|\eta|}{\gamma}.} \tag{55}$$

Thus a acts as a scanning parameter: $1 - a \asymp \gamma^{-1}$ selects a high-zero window of height comparable to γ. The statement concerns the conformal growth exponent of the recentered family; finite coefficient onset still depends on the transformed residue and background.

Proof. From

$$|w_\rho|^2 = \frac{e^{-2g_\rho} - 2ae^{-g_\rho}\cos\phi_\rho + a^2}{1 - 2ae^{-g_\rho}\cos\phi_\rho + a^2e^{-2g_\rho}},$$

multiplying numerator and denominator by e^{g_ρ} and writing the result in hyperbolic functions gives

$$\frac{|1-av_\rho|^2}{|v_\rho - a|^2} = \frac{A+B}{A-B}, \quad A = (1+a^2)\cosh g_\rho - 2a\cos\phi_\rho, \quad B = (1-a^2)\sinh g_\rho.$$

Taking one half of the logarithm proves (52); differentiating the exact identity at $g = 0$ gives (53).

Differentiating $P_a(\phi)$ with respect to a gives a numerator proportional to

$$a^2\cos\phi - 2a + \cos\phi.$$

For $0 < |\phi| < \pi/2$, the root in $(0,1)$ is (54), and direct substitution gives $P_{a_*} = 1/|\sin\phi|$. Finally, the complex angle determined by $1 - 2\vartheta_\rho = \cos\alpha_\rho$ satisfies

$$\alpha_\rho = 2\sqrt{\vartheta_\rho} + O(|\vartheta_\rho|^{3/2}),$$

so for bounded η,

$$|\phi_\rho| = |\operatorname{Re}\alpha_\rho| = \frac{\sqrt{27}}{\gamma} + O(\gamma^{-3}), \qquad g_\rho = |\operatorname{Im}\alpha_\rho| = \frac{\sqrt{27}|\eta|}{\gamma^2}\Big(1 + O(\gamma^{-2})\Big).$$

At $a = a_*(\phi_\rho)$ one has $P_a(\phi_\rho) = 1/|\sin\phi_\rho|$, while $g_\rho/|\phi_\rho| \to 0$. The exact formula (52) therefore gives

$$-\log|w_\rho| = \frac{g_\rho}{|\sin\phi_\rho|}(1+o(1)) \sim \frac{|\eta|}{\gamma},$$

which proves (55). □

Remark 6.10 (Conformal amplification versus finite locality)*.* The Möbius family recovers one power of height, but it changes the information model. When $a \neq 0$, the Taylor coefficients of $\mathcal{C}_a$ at 0 depend on derivatives of $\mathcal{C}$ at the interior point a and hence, in general, on infinitely many original trace coefficients rather than on a finite set $h_0, \dots, h_n$. Thus the scanning family exhibits a precise tradeoff:

$$\begin{gathered}\text{finite three-point locality with } |\eta|/\gamma^2 \text{ Green exponent}\\ \longleftrightarrow\\ \text{scale-dependent recentering with } |\eta|/\gamma \text{ Green exponent.}\end{gathered}$$

The improvement is therefore real, but it is purchased by leaving the finite-local class of Theorem 6.6.

7. Carathéodory, Schur, Clark measure, and a Hermite–Biehler lift

The bounded trace is the coefficient shadow of a positive-real analytic function.

Theorem 7.1 (Carathéodory form)**.** *The following are equivalent:*

(1) *RH;*
(2) *$\mathcal{C}$ is holomorphic on $\mathbb{D}$ and $\operatorname{Re}\mathcal{C}(v) > 0$ for $v \in \mathbb{D}$;*
(3) *the normalized function $H(v) = \mathcal{C}(v)/h_0$ is Carathéodory on $\mathbb{D}$;*
(4)

$$S(v) = \frac{H(v) - 1}{H(v) + 1} \tag{56}$$

is Schur on $\mathbb{D}$, i.e. $|S(v)| \le 1$ for $v \in \mathbb{D}$.

On RH,

$$\mathcal{C}(v) = \sum_\gamma \vartheta_\gamma \frac{1 - v^2}{1 - 2v\cos\phi_\gamma + v^2}, \tag{57}$$

and $S(0) = 0$.

Proof. Under RH, $y_\gamma = \cos\phi_\gamma$. From (37) and the cosine generating series,

$$\mathcal{C}(v) = \sum_\gamma \frac{\vartheta_\gamma}{2}\left(\frac{1 + ve^{i\phi_\gamma}}{1 - ve^{i\phi_\gamma}} + \frac{1 + ve^{-i\phi_\gamma}}{1 - ve^{-i\phi_\gamma}}\right),$$

which has positive real part in $\mathbb{D}$. The Carathéodory–Schur equivalence is classical. Conversely, either (2), (3), or (4) implies the Carathéodory coefficient bound $|u_n| \le h_0$; Theorem 6.3 then gives RH. □

The Schur function carries the original cubic node measure without loss. To compare with the companion positivity coordinates, write

$$D_{r,k} := \sum_{\rho} \vartheta_\rho^{r+1}(1-\vartheta_\rho)^k, \qquad b_k := D_{0,k}, \tag{58}$$

which is the two-index beta lattice (Pascal form) of [10].

Proposition 7.2 (Clark measure and the beta–trace bridge)**.** *Assume RH and define the probability measure on the unit circle*

$$d\sigma(\zeta) = \frac{1}{2h_0}\sum_{\gamma>0} \mathrm{mult}(\gamma)\vartheta_\gamma \left(\delta_{e^{i\phi_\gamma}} + \delta_{e^{-i\phi_\gamma}}\right), \qquad \cos\phi_\gamma = 1 - 2\vartheta_\gamma. \tag{59}$$

Then

$$H(v) = \int_{\mathbb{T}} \frac{\zeta+v}{\zeta-v}\, d\sigma(\zeta), \qquad H = \frac{1+S}{1-S}. \tag{60}$$

Thus σ is the Aleksandrov–Clark measure of S at parameter 1. Since σ is purely atomic, S is inner.

Moreover the circle trace and the full two-index beta lattice (Pascal form) are two coordinate systems of the same measure (with $\zeta = e^{i\phi}$ in the following integrals):

$$\frac{u_n}{h_0} = \int_{\mathbb{T}} \cos(n\phi)\, d\sigma(\zeta), \tag{61}$$

$$\frac{D_{r,k}}{h_0} = \int_{\mathbb{T}} \left(\frac{1-\cos\phi}{2}\right)^r \left(\frac{1+\cos\phi}{2}\right)^k d\sigma(\zeta). \tag{62}$$

In particular,

$$\frac{b_k}{h_0} = \int_{\mathbb{T}} \left(\frac{1+\cos\phi}{2}\right)^k d\sigma(\zeta). \tag{63}$$

Proof. Equation (60) is the Herglotz representation obtained by dividing the symmetric atomic formula in the proof of Theorem 7.1 by h_0. Since $S = (H-1)/(H+1)$, this is the Clark representation at parameter 1 [2]. A purely singular Herglotz measure gives unimodular radial boundary values for S almost everywhere, hence S is inner. Equation (61) is immediate from (30). For (62), use

$$\vartheta = \frac{1-\cos\phi}{2}, \qquad 1-\vartheta = \frac{1+\cos\phi}{2},$$

and the node expansion $D_{r,k} = \sum_\gamma \vartheta_\gamma^{r+1}(1-\vartheta_\gamma)^k$. □

The bridge can also be written as a finite linear transform. Let

$$p_{r,k}(x) = 2^{-(r+k)}(1-x)^r(1+x)^k = \sum_{n=0}^{r+k} c_n^{(r,k)} T_n(x).$$

Then

$$\boxed{D_{r,k} = \sum_{n=0}^{r+k} c_n^{(r,k)} u_n.} \tag{64}$$

The coefficients are explicit:

$$c_n^{(r,k)} = \frac{2-\delta_{n0}}{\pi}\int_0^\pi \sin^{2r}\left(\frac{\phi}{2}\right)\cos^{2k}\left(\frac{\phi}{2}\right)\cos(n\phi)\, d\phi. \tag{65}$$

Thus the two-index beta lattice (Pascal form) is the Bernstein face and the trace is the Fourier–Chebyshev face of one measure. In particular, the unconditional positive wedge proved for $D_{r,k}$

in the companion cubic-descent paper [10] transfers to an explicit family of positive finite linear forms in the trace coefficients through (64). It does not by itself imply coefficientwise boundedness; it identifies precisely how the two newest coordinate systems encode the same partial theorem.

The same statement becomes especially simple in a half-plane variable. Put

$$z = \frac{1+v}{1-v}, \qquad G(z) = K(c(z^2-1)). \tag{66}$$

Then $\operatorname{Re} z > 0$ corresponds to $v \in \mathbb{D}$.

Proposition 7.3 (Positive-real logarithmic derivative)**.** *For $v = (z-1)/(z+1)$,*

$$\boxed{\mathcal{C}(v) = \frac{1}{2}\frac{G'(z)}{G(z)}.} \tag{67}$$

Consequently

$$\boxed{\text{RH} \iff \operatorname{Re}\frac{G'(z)}{G(z)} > 0 \quad (\operatorname{Re} z > 0).} \tag{68}$$

Proof. From (66),

$$c(z^2-1) = \frac{4cv}{(1-v)^2} = X(v).$$

Also

$$\frac{G'(z)}{G(z)} = 2cz\frac{K'(X(v))}{K(X(v))}.$$

Since $(1-v^2)/(4v) = z/(z^2-1)$, (38) gives (67). The equivalence follows from Theorem 7.1. □

This produces a canonical Hermite–Biehler-type object. Set

$$E(z) = G'(z) + 2h_0G(z). \tag{69}$$

Since G is even and real entire,

$$E(-z) = -G'(z) + 2h_0G(z).$$

Thus

$$S\left(\frac{z-1}{z+1}\right) = -\frac{E(-z)}{E(z)}. \tag{70}$$

Corollary 7.4 (Right-half-plane Hermite–Biehler form)**.** *RH is equivalent to*

$$|E(-\bar{z})| \le |E(z)| \qquad (\operatorname{Re} z > 0), \tag{71}$$

with strict inequality away from degenerate boundary limits.

Proof. Because E has real coefficients, $|E(-\bar{z})| = |E(-z)|$. Now use (70) and Theorem 7.1. □

Remark 7.5. Hermite–Biehler and de Branges approaches to RH have a substantial literature; see, for example, Suzuki [12]. Corollary 7.4 is not presented as the first Hermite–Biehler criterion. Its role here is that the Hermite–Biehler inequality emerges canonically from the cubic trace, with E built from the descended K and the local mass h_0.

Corollary 7.6 (Canonical de Branges object)**.** *Put*

$$\mathcal{E}(w) = E(-iw), \qquad \mathcal{E}^{\#}(w) = \overline{\mathcal{E}(\bar{w})}.$$

Then RH is equivalent to $\mathcal{E}$ being Hermite–Biehler in the upper half-plane:

$$|\mathcal{E}^{\#}(w)| < |\mathcal{E}(w)| \qquad (\operatorname{Im} w > 0).$$

On RH, $\mathcal{E}$ therefore generates a canonical de Branges space $\mathcal{H}(\mathcal{E})$, and the associated inner quotient satisfies

$$-\frac{\mathcal{E}^{\#}(w)}{\mathcal{E}(w)} = S\left(\frac{-iw-1}{-iw+1}\right).$$

Thus the Clark measure in Proposition 7.2 and the Hermite–Biehler lift belong to the same inner function.

Proof. Because E is real entire, $\mathcal{E}^{\#}(w) = E(iw)$. The half-plane $\operatorname{Im} w > 0$ is carried by $z = -iw$ to $\operatorname{Re} z > 0$, so Corollary 7.4 gives the strict Hermite–Biehler inequality. The quotient identity follows from (70). The existence of the de Branges space is then classical. If $\mathcal{E}$ and $\mathcal{E}^{\#}$ have a common real factor (which can occur when a descended zero has multiplicity greater than one), that common factor may be divided out first; the inner quotient and the Hermite–Biehler inequality are unchanged after the resulting removable cancellation. □

8. A Wiener variance law

Boundedness can in principle hide phase cancellation along selected indices. Cesàro averages of squares do not.

Merge equal node values on RH and write the distinct nodes as ν, with multiplicity m_ν and weight

$$a_\nu = m_\nu \nu.$$

Define

$$V_N = \frac{2}{N}\sum_{n=1}^{N} u_n^2. \tag{72}$$

Theorem 8.1 (Variance dichotomy)**.** *If RH holds, then*

$$\lim_{N\to\infty} V_N = \sum_\nu a_\nu^2. \tag{73}$$

Moreover

$$h_1 \le \sum_\nu a_\nu^2 \le h_0^2, \tag{74}$$

and equality on the left holds exactly when every transported node is simple.

If RH fails, then

$$\boxed{\lim_{N\to\infty} \frac{1}{2N}\log V_N = \sup_\rho g(y_\rho) > 0.} \tag{75}$$

Proof. On RH, symmetrize the measure in (30) to an even measure on the unit circle. Its Fourier coefficients are u_n. Wiener's theorem gives the Cesàro limit as the sum of squares of the point masses, which is (73). Since

$$\sum_\nu a_\nu = h_0, \qquad \sum_\nu m_\nu \nu^2 = h_1,$$

we have

$$\sum_\nu a_\nu^2 = \sum_\nu m_\nu^2 \nu^2 \ge \sum_\nu m_\nu \nu^2 = h_1,$$

with equality exactly when $m_\nu = 1$ for every node. Since every $a_\nu \ge 0$,

$$\sum_\nu a_\nu^2 \le \left(\sum_\nu a_\nu\right)^2 = h_0^2,$$

which is the upper bound in (74).

If RH fails, let $Z = e^{\sup g} > 1$. By Lemma 6.1, only finitely many merged node values have $|\zeta_*(y)| = Z$. After equal node values are merged, list the distinct maximal exterior phases as

$$\zeta_j = Z\lambda_j, \qquad |\lambda_j| = 1, \qquad j = 1, \dots, M,$$

including conjugate phases as separate entries when they are distinct, with nonzero coefficients c_j. The reciprocal branches and all nodes of smaller Green depth are exponentially subordinate, so

$$Z^{-n} u_n = \sum_{j=1}^{M} c_j \lambda_j^n + o(1). \tag{76}$$

The phases λ_j are distinct by construction. Hence the $M \times M$ Vandermonde matrix

$$V = (\lambda_j^k)_{0 \le k \le M-1,\ 1 \le j \le M}$$

is invertible. Multiplication of the coefficient vector (c_j) by the diagonal unitary $\operatorname{diag}(\lambda_j^n)$ does not change its norm, so there is a constant $c_* > 0$ such that, for every n,

$$\sum_{k=0}^{M-1} \left| \sum_{j=1}^{M} c_j \lambda_j^{n+k} \right|^2 \ge c_*.$$

Using (76), the last M terms before N therefore give a lower bound $\gg Z^{2N}$ for $\sum_{n \le N} u_n^2$, while the elementary upper bound from the finitely many maximal modes and the subordinate tail is $\ll Z^{2N}$. Consequently

$$\sum_{n=1}^{N} u_n^2 = Z^{2N+o(N)}, \qquad V_N = Z^{2N+o(N)},$$

which gives (75). □

Corollary 8.2 (Multiplicity excess)**.** *On RH,*

$$\boxed{\lim_{N\to\infty} V_N - h_1 = \sum_{\nu} m_\nu (m_\nu - 1)\nu^2 \ge 0.} \tag{77}$$

Thus the excess of the Wiener energy over the local moment h_1 is exactly the collision/multiplicity energy of the descended nodes. It vanishes precisely when every node value occurs once.

Proof. Subtract $h_1 = \sum_\nu m_\nu \nu^2$ from (73). □

Remark 8.3. The quantity in (77) is recoverable from local data through the Cesàro limit of the locally defined trace. It is not a new finite formula for zero multiplicities; rather, it isolates exactly what the phase-robust trace average measures beyond the first local moment.

Remark 8.4. The asymptotic law (75) is phase-robust. A separate argument is required before turning it into a uniform finite-N zero-free lens, because cross terms can be negative before the maximal exponential class dominates. We therefore do not claim a finite verification theorem here.

9. Trace filters on the von Mangoldt measure

The trace remains connected to primes because its local residue is taken against $\ell = \xi'/\xi$. Split

$$\ell(s) = \ell_\infty(s) + \frac{\zeta'(s)}{\zeta(s)}, \tag{78}$$

where

$$\ell_\infty(s) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right).$$

Definition 9.1 (Trace prime filter)**.** For $n \geq 0$ and $x \geq 0$, define

$$\Pi_n(x) = -\operatorname*{Res}_{s=2} e^{-x(s-2)} W(s) T_n(1 + 2W(s)). \tag{79}$$

Define the completed non-prime term explicitly by

$$A_n^{\mathrm{tr}} := \operatorname*{Res}_{s=2} \ell_\infty(s) W(s) T_n(1 + 2W(s)) + \frac{1}{2} \operatorname*{Res}_{s=1/2} \ell(s) W(s) T_n(1 + 2W(s)). \tag{80}$$

In particular,

$$A_0^{\mathrm{tr}} = \frac{4}{3}(\ell_\infty(2) + \ell'_\infty(2)) + \frac{1}{24}\ell'(1/2) = \frac{1}{3} - \frac{2}{3}(\gamma_E + \log \pi) + \frac{\pi^2}{18} + \frac{1}{24}\ell'(1/2), \tag{81}$$

where γ_E is Euler's constant. Numerically,

$$A_0^{\mathrm{tr}} = -0.26439359545826163 01\ldots.$$

Thus the term A_n^{tr} in the completed trace bound is a fully specified local residue functional, not an undetermined remainder.

Theorem 9.2 (Prime-filter decomposition of the trace)**.** *For every $n \geq 0$,*

$$\boxed{u_n = A_n^{\mathrm{tr}} + \sum_{m \geq 2} \frac{\Lambda(m)}{m^2} \Pi_n(\log m).} \tag{82}$$

The prime sum converges absolutely. The function Π_n is a polynomial of degree $2n + 1$ and

$$\int_0^\infty e^{-x} \Pi_n(x)\, dx = 0. \tag{83}$$

Moreover Π_n lies in the finite odd generalized-Laguerre sector

$$\operatorname{span}\{L_1^{(1)}(2x), L_3^{(1)}(2x), \ldots, L_{2n+1}^{(1)}(2x)\}.$$

Proof. For $\operatorname{Re} s > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{m \geq 2} \Lambda(m) m^{-s}.$$

Insert $m^{-s} = m^{-2} e^{-(s-2)\log m}$ into the $s = 2$ residue of (29); the ℓ_∞ residue at 2 together with the full central residue is exactly (80), and the Dirichlet-series part gives (82). Since W has a double pole at 2 and T_n has degree n, the residue extraction yields a polynomial of degree $2n + 1$.

For (83), integrate (79) against e^{-x} and obtain the residue at $s = 2$ of

$$-\frac{W(s) T_n(1 + 2W(s))}{s - 1}.$$

The involution $s \mapsto s/(s-1)$ fixes 2, leaves W invariant, and reverses the differential $ds/(s-1)$; the residue equals its own negative and vanishes. The Laguerre statement follows because u_n is an integer combination of the original moment filters, each of which belongs to the odd generalized-Laguerre sector [10]. □

The first filters are

$$\Pi_0(x) = \frac{4}{3}(x - 1), \tag{84}$$

$$\Pi_1(x) = \frac{4}{81}\left(12x^3 - 72x^2 + 71x + 1\right), \tag{85}$$

$$\Pi_2(x) = \frac{4}{3645}\left(144x^5 - 2160x^4 + 9120x^3 - 11520x^2 + 2935x - 55\right). \tag{86}$$

Each has zero e^{-x}-mean, as required.

At $n = 0$ the completed decomposition has a zero-free consistency check that involves no zeta ordinates. From (84),

$$h_0 - A_0^{\mathrm{tr}} = \frac{4}{3}\left[\frac{\zeta''}{\zeta}(2) - \left(\frac{\zeta'}{\zeta}(2)\right)^2 + \frac{\zeta'}{\zeta}(2)\right] = 0.41936112115865477173\ldots. \tag{87}$$

Together with (14) and (81), this checks the local normalization, the completed split, and the first prime filter against one another with no free parameter.

Combining Theorems 6.3 and 9.2 gives an arithmetic form of the remaining problem:

$$\boxed{\mathrm{RH} \iff \left| A_n^{\mathrm{tr}} + \sum_{m\geq 2} \frac{\Lambda(m)}{m^2}\Pi_n(\log m)\right| \leq h_0 \quad \text{for every } n \geq 0.} \tag{88}$$

This is not a proof: the completed prime–archimedean bound remains the unresolved step. It does, however, place the whole criterion in one one-dimensional family of explicit zero-mean logarithmic filters. The prime block by itself should not be expected to carry the sign; the earlier cubic moment filters already exhibit an exact Fourier obstruction to prime-only positivity [10].

10. Numerical illustration

The trace can be evaluated on the local side, without supplying zero ordinates. Using the three-point residue formula gives

$$h_0 = 0.154967525700\ldots \tag{89}$$

and

n	u_n
0	$0.154967525700\ldots$
1	$0.151682789679\ldots$
2	$0.142164397047\ldots$
3	$0.127379013740\ldots$
4	$0.108807040262\ldots$
5	$0.088264152797\ldots$

A zero-side sum over the first 1000 ordinates differs from these local values by approximately 0.00486 across these small indices. This is quantitatively accounted for by the missing positive tail. Since

$$\vartheta_\gamma = \frac{c}{\gamma^2}\left(1 + O(\gamma^{-2})\right)$$

and $dN(t) \sim (2\pi)^{-1}\log(t/2\pi)\,dt$ from the Riemann–von Mangoldt law [13], the leading tail estimate is

$$\sum_{\gamma > T} \vartheta_\gamma \sim \frac{c}{2\pi T}\left(\log\frac{T}{2\pi} + 1\right). \tag{90}$$

At $T = \gamma_{1000} = 1419.4224809\ldots$, the right side is

$$0.00485910\ldots,$$

matching the observed 0.00486 gap at the displayed precision. The same effect is nearly index-independent at fixed truncation, as the large nodes satisfy $T_n(y_\gamma) = 1 + O(n^2/\gamma^2)$ for fixed n. For example, truncating at the first 400 ordinates gives local-minus-zero-side gaps

$$0.0089743,\ 0.0089742,\ 0.0089740,\ 0.0089736,\ 0.0089731,\ 0.0089724$$

for $n = 0, \ldots, 5$, while (90) gives $0.0089830\ldots$ at $T = \gamma_{400}$. This flatness is a second diagnostic of the same missing-tail mechanism, not an input to any proof.

For a synthetic off-line pair at $(\gamma, \eta) = (88.26, 0.4)$, the measured Green escape is approximately 2.667×10^{-4}, while (47) predicts 2.668×10^{-4}. The trace remains bounded-looking for a long transient and then grows exponentially. This is consistent with the distinction in Remark 6.7: Theorem 6.3 fixes the asymptotic exponent, while the small node weight and the remaining background affect the first visible crossing. These calculations are diagnostic only; none enters the proof of the equivalence.

11. Context, contribution, and the remaining problem

The progression can now be stated without ambiguity:

$$\begin{aligned} \text{anharmonic orbit of } \xi \longrightarrow q^2 \longrightarrow K \longrightarrow (m_n) \text{ Hankel} \longrightarrow (h_n) \text{ Hausdorff} \\ \longrightarrow J \text{ Jacobi} \longrightarrow D_{r,k} \text{ two-index beta lattice (Pascal form)} \\ \longrightarrow b_k \text{ boundary} \longrightarrow u_n \text{ Chebyshev trace} \\ \longrightarrow \mathcal{C} \text{ Carathéodory} \longrightarrow S \text{ Schur.} \end{aligned}$$

The positivity coordinates are classical moment-theoretic faces of the same descended zero set. Proposition 4.2 shows that the trace step is a lossless finite-level basis change. The genuinely different mechanism begins with the Koebe factorization: a zero of K is pulled back to a trace pole, an off-ray preimage lies inside the disk, and Lemma 6.2 proves that it cannot cancel. Cauchy–Hadamard then turns horizontal zero motion into exponential coefficient growth. That is why boundedness, normally far weaker than positivity, is equivalent to the full zero-location statement in this rigid family.

The second structural result is not another equivalence. Theorem 6.6 is an optimality statement in a specified information class: among normalized degree-n polynomial probes that remain finite combinations of the three-point moments, no probe can beat the Bernstein–Walsh exponent ng, while Chebyshev reaches that exponent up to an absolute factor. Proposition 6.9 prices an explicit way to improve the conformal exponent: scale-dependent recentering gains one power of height but leaves the finite-local information model.

The classical ingredients are credited as such. Hamburger and Hausdorff moment theory are classical [1]; the λ-to-j identity is classical [3]; Chebyshev and Bernstein–Walsh belong to classical approximation theory [9]; Carathéodory, Schur, Clark, Wiener, and Hermite–Biehler theory are classical [2, 11, 6, 12]. Li's criterion is an exact scalar positivity sequence [7]; Jensen-polynomial criteria provide a different hyperbolicity coordinate [5, 8]; and Hausdorff/genus-zero criteria have also been applied directly to RH-type functions [14]. We do not claim that the present criterion dominates any of these or is "closer" to a proof.

The contribution claimed here is narrower and mechanistic: the order-three descent and its single exceptional residue orbit admit a lossless Chebyshev coordinate in which the reverse RH implication is governed by non-cancellable Koebe poles; the earlier two-index beta lattice (Pascal form) and the trace become two coordinate systems of one Clark measure; and the finite-local class admits an exact amplification frontier with an explicit price for conformal recentering.

No theorem above proves the final arithmetic inequality. It can now be asked in one line.

Problem 11.1 (Completed trace bound)**.** Prove directly from the completed prime–archimedean decomposition (82) that

$$|u_n| \le h_0 \qquad (n \ge 0),$$

or equivalently prove that

$$H(v) = \frac{1}{h_0} \frac{1 - v^2}{4v} F\left(\frac{27v}{(1-v)^2}\right)$$

has positive real part on $\mathbb{D}$, or that its Cayley transform (56) is Schur.

The advantage of this formulation is precise rather than rhetorical: if the bound fails because RH fails, failure is not a hidden determinant sign but an explicit interior singularity, and its exponential coefficient rate is exactly the conformal Green depth of the offending transported zero. A solution of Problem 11.1 would add genuinely new arithmetic information after the change of coordinates and would imply RH. The present paper does not supply that inequality.

Acknowledgments

This work is the culmination of years of questioning the nature and distribution of prime numbers, facing Dunning–Kruger head on, and consistently pushing past our failures. All we wanted was to solve the Riemann Hypothesis. Instead, we learned what true rigor and humility really are.

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