A BOUNDARY-LAYER CALCULUS FOR CHEBYSHEV PRIME FILTERS

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ABSTRACT. We isolate a local scaling phenomenon associated with a Chebyshev trace construction for the Riemann xi function. Near the exceptional point $s = 2$, the orbital weight $W(s) = 27/(4q(s)^2)$ has a double pole. This forces the ordinary Chebyshev generating radius to collapse. After the diagonal scaling $v = \lambda t^2$, $x = \mu/t$, $t = s - 2$, the collapse resolves into two distinct microscopic objects depending on the order of operations. Formal summation before residue extraction gives a geometric resolvent with critical value $\lambda = 3/16$, whereas legal residue extraction first produces factorially regularized prime-filter diagonals. The leading diagonal sums to a hyperbolic-sine master function. Successive diagonals are obtained from it by polynomial differential operators in the dilation generator $\mathcal{D} = (z/2)\partial_z$. We give the first five scaling functions and an exact coefficient mechanism generating all further diagonals. The result is local and asymptotic; no claim toward a proof of the Riemann hypothesis is made.

1. Introduction

The Chebyshev trace attached to the cubic descent of the Riemann xi function [1] admits a finite-local arithmetic description: for each index n, its prime contribution is encoded by a polynomial filter $\Pi_n(\log m)$. The filters are individually elementary to define, but their degrees grow with n. Near the exceptional point $s = 2$, the weight from which they are extracted has a double pole, while the radius of the ordinary Chebyshev generating series collapses to zero. The purpose of this note is to describe what remains when one looks at that collapse at its natural scale.

The answer is unexpectedly rigid. There are two natural orders of operation, and they lead to two different microscopic objects. Formal Chebyshev resummation before taking the local residue retains a geometric singularity at a finite scaled coordinate. Taking the residue first—the order in which the prime filters are actually defined—introduces a factorial denominator and converts the same geometric hierarchy into an entire hyperbolic master function. The successive coefficient diagonals are then generated from that one function by polynomials in a dilation operator.

The simplest way to state the phenomenon is this: *the local residue does not merely extract the Chebyshev hierarchy; it regularizes it.* The regularized seed is not an unnamed special function: it is the two-parameter Mittag–Leffler function $E_{2,2}$, since

$$E_{2,2}(w) = \sum_{n\geq 0} \frac{w^n}{\Gamma(2n+2)}.$$

Thus the odd factorials produced by the residue place the boundary layer naturally in the Mittag–Leffler family [2]. Geometric growth in the mode number is transformed into factorially damped growth, and the resulting filter triangle is organized by differential descendants of

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$\sinh z/z$. This note records that local mechanism and nothing more. In particular, no hypothesis on the zeros of ζ is used, and no claim toward a proof of the Riemann hypothesis is made.

2. Setup and statement of the local result

Let

$$q(s) = \frac{(s-2)(s+1)(2s-1)}{2s(s-1)}, \qquad \kappa = \frac{27}{4}, \qquad W(s) = \frac{\kappa}{q(s)^2}.$$

These quantities arise from the order-three quotient used in the Chebyshev trace construction [1]. Put $t = s-2$. Direct Taylor expansion gives

$$q(2+t) = \frac{9}{4}t - \frac{9}{8}t^2 + \frac{17}{16}t^3 - \frac{33}{32}t^4 + O(t^5), \tag{1}$$

$$W(2+t) = \frac{4}{3t^2} + \frac{4}{3t} - \frac{7}{27} + O(t^2). \tag{2}$$

It is convenient to write

$$W(2+t) = \frac{4}{3t^2}a(t), \qquad a(t) = 1 + t - \frac{7}{36}t^2 + 0\cdot t^3 + \frac{1}{27}t^4 + O(t^5). \tag{3}$$

The vanishing cubic coefficient is exact; it is not a truncation convention. The displayed t^4-term is included because it is the first additional jet needed to reproduce the fourth diagonal below.

For $n \geq 0$, let T_n be the Chebyshev polynomial of the first kind. Define the local prime filter

$$\Pi_n(x) := -\operatorname*{Res}_{s=2} e^{-x(s-2)}W(s)T_n(1+2W(s)). \tag{4}$$

For fixed n, Π_n is a polynomial of degree $2n+1$. The first examples are

$$\begin{aligned}
\Pi_0(x) &= \frac{4}{3}x - \frac{4}{3},\\
\Pi_1(x) &= \frac{16}{27}x^3 - \frac{32}{9}x^2 + \frac{284}{81}x + \frac{4}{81},\\
\Pi_2(x) &= \frac{64}{405}x^5 - \frac{64}{27}x^4 + \frac{2432}{243}x^3 - \frac{1024}{81}x^2 + \frac{2348}{729}x - \frac{44}{729}.
\end{aligned}$$

All statements below concern this local family.

Theorem 1 (Boundary-layer master calculus)**.** *Let Π_n be defined by (4), and write*

$$\Pi_n(x) = \sum_{j=0}^{2n+1} c_{n,j}x^{2n+1-j}.$$

For $n \geq 1$,

$$c_{n,0} = \frac{2^{4n+1}}{3^{n+1}(2n+1)!}.$$

For each fixed diagonal $j \geq 0$, the ratio $c_{n,j}/c_{n,0}$, on the range where that diagonal occurs, is determined by the finite coefficient formula (25); in particular it is a polynomial $p_j(n)$. Under the diagonal scaling

$$x = \mu/t, \qquad v = \lambda t^2, \qquad t = s-2,$$

the residue-first generating family has the coefficientwise asymptotic expansion

$$t\sum_{n\geq 1}\Pi_n(\mu/t)(\lambda t^2)^n \sim \sum_{j\geq 0} t^j g_j(\lambda,\mu),$$

where, with

$$z = 4\mu\sqrt{\lambda/3}, \qquad \mathcal{D} = \frac{z}{2}\frac{d}{dz},$$

one has

$$g_0 = \frac{2\mu}{3}\left(\frac{\sinh z}{z} - 1\right), \qquad g_j = \mu^{-j} p_j(\mathcal{D}) g_0.$$

By contrast, formal Chebyshev summation before residue extraction has the scaled boundary resolvent, with F *as in* (5),

$$\lim_{t\to 0} t^2 F(2+t, \lambda t^2) = \frac{64\lambda}{3(16\lambda - 3)},$$

with critical value $\lambda_c = 3/16$.

Here and below, "coefficientwise" means that for each fixed power of λ the asserted asymptotic expansion is the ordinary finite Laurent calculation coming from the corresponding polynomial Π_n. No interchange of an unproved infinite asymptotic expansion is being assumed.

3. The collapsing Chebyshev radius

The classical generating identity

$$\sum_{n\geq 1} T_n(Y) v^n = \frac{v(Y - v)}{1 - 2Yv + v^2}$$

with $Y = 1 + 2W$ yields the formal kernel

$$F(s, v) = \frac{2W(s)v\big(v - 1 - 2W(s)\big)}{(1-v)^2 - 4vW(s)}. \tag{5}$$

Because $W(2+t) \sim 4/(3t^2)$, the convergence radius in v tends to zero as $t \to 0$. Thus the infinite Chebyshev sum cannot in general be interchanged with the residue in (4).

Nevertheless the boundary layer is resolved by

$$v = \lambda t^2.$$

Indeed,

$$(1-v)^2 - 4vW(2+t) = 1 - \frac{16}{3}\lambda + O(t),$$

so the scaled coordinate has the exact critical value

$$\boxed{\lambda_c = \frac{3}{16}}. \tag{6}$$

Moreover, for $\lambda \neq 3/16$,

$$\lim_{t\to 0} t^2 F(2+t, \lambda t^2) = \frac{64\lambda}{3(16\lambda - 3)}. \tag{7}$$

Writing $r = 16\lambda/3$, this is $-(4/3)r/(1-r)$: a geometric resolvent with a finite pole at $r = 1$.

4. Residue-first scaling and the master function

The meaningful comparison is obtained by respecting the residue in (4) first. Write

$$\Pi_n(x) = \sum_{j=0}^{2n+1} c_{n,j} x^{2n+1-j}. \tag{8}$$

For $n \geq 1$, the highest-degree coefficient is

$$\boxed{c_{n,0} = \frac{2^{4n+1}}{3^{n+1}(2n+1)!}}. \tag{9}$$

To see this, use $T_n(y) = 2^{n-1}y^n + \cdots$ and $W \sim 4/(3t^2)$, so

$$WT_n(1+2W) \sim 2^{2n-1}\left(\frac{4}{3}\right)^{n+1} t^{-2n-2}.$$

The residue is supplied by the t^{2n+1} term of e^{-xt}, giving (9). The case $n = 0$ is exceptional only because $T_0 = 1$, so its leading Chebyshev coefficient is not 2^{n-1}.

Now impose the two-variable scaling

$$x = \frac{\mu}{t}, \qquad v = \lambda t^2, \tag{10}$$

and define, coefficientwise in λ,

$$\mathcal{G}(t;\lambda,\mu) := t\sum_{n\geq 1}\Pi_n(\mu/t)(\lambda t^2)^n. \tag{11}$$

Then

$$\mathcal{G}(t;\lambda,\mu) \sim \sum_{j\geq 0} t^j g_j(\lambda,\mu), \tag{12}$$

where

$$g_j(\lambda,\mu) = \sum_{n\geq 1} c_{n,j}\lambda^n\mu^{2n+1-j}.$$

Set

$$z = 4\mu\sqrt{\frac{\lambda}{3}}. \tag{13}$$

Summing (9) gives the master function

$$\boxed{g_0(\lambda,\mu) = \frac{2\mu}{3}\left(\frac{\sinh z}{z} - 1\right) = \frac{\sinh z}{2\sqrt{3\lambda}} - \frac{2\mu}{3}.} \tag{14}$$

Equivalently,

$$\boxed{g_0(\lambda,\mu) = \frac{2\mu}{3}\big(E_{2,2}(z^2) - 1\big),} \qquad E_{2,2}(w) = \sum_{n\geq 0}\frac{w^n}{\Gamma(2n+2)}. \tag{15}$$

The elementary identity $E_{2,2}(z^2) = \sinh z/z$ makes the Mittag–Leffler structure explicit. Thus residue extraction replaces the geometric growth of the sum-first boundary layer by factorial suppression $(2n+1)!^{-1}$. In this precise local sense, the residue-first operation factorially regularizes the Chebyshev boundary layer.

5. Diagonal differential calculus

Define the dilation operator

$$\mathcal{D} = \frac{z}{2}\frac{d}{dz}. \tag{16}$$

Since $\mathcal{D}(z^{2n}) = nz^{2n}$, polynomial dependence on the filter index becomes differential dependence on z.

For the first five diagonals one has

$$c_{n,j} = p_j(n)c_{n,0}, \tag{17}$$

where

$$
\begin{aligned}
p_0(n) &= 1,\\
p_1(n) &= -(n+1)(2n+1),\\
p_2(n) &= \frac{n(2n+1)(36n^2+49n-14)}{36},\\
p_3(n) &= -\frac{n^2(2n-1)(2n+1)(12n^2+13n-26)}{36},\\
p_4(n) &= \frac{n(n-1)(2n-1)(2n+1)}{5184}\\
&\qquad \times (864n^4+144n^3-4414n^2+2717n+768).
\end{aligned}
$$

Consequently

$$\boxed{g_j(\lambda,\mu) = \mu^{-j} p_j(\mathcal{D}) g_0(\lambda,\mu).} \tag{18}$$

The apparent negative powers of μ are removable in the power-series interpretation of (12).

Applying (18) yields

$$g_1 = \frac{2}{3}(1-\cosh z) - \frac{z}{3}\sinh z, \tag{19}$$

$$g_2 = \frac{z\big(18z^2\sinh z + 103z\cosh z + 39\sinh z\big)}{216\mu}, \tag{20}$$

$$g_3 = -\frac{z^2\big((67z^2-4)\cosh z + (6z^3+127z)\sinh z\big)}{432\mu^2}, \tag{21}$$

$$
\begin{aligned}
g_4 = \frac{z^3}{62208\mu^3}\Big(&(1980z^3+7500z)\cosh z\\
&+ (108z^4+8701z^2-456)\sinh z\Big).
\end{aligned} \tag{22}
$$

Proposition 1 (Hyperbolic closure)**.** *For every diagonal for which p_j is defined, $g_j = \mu^{-j}p_j(\mathcal{D})g_0$ belongs to*

$$\mathbb{Q}(z,\mu) + \mathbb{Q}(z,\mu)\sinh z + \mathbb{Q}(z,\mu)\cosh z.$$

Proof. The operator $\mathcal{D} = (z/2)d/dz$ preserves this space, since differentiation interchanges $\sinh z$ and $\cosh z$ and multiplication by z preserves rational coefficients. The claim follows from (18). □

Thus the five displayed functions are the first explicit members of a closed hyperbolic differential family, rather than evidence for closure based only on low-order computation.

As checks, the expansions begin

$$
\begin{aligned}
g_0 &= \frac{16}{27}\lambda\mu^3 + \frac{64}{405}\lambda^2\mu^5 + \cdots ,\\
g_1 &= -\frac{32}{9}\lambda\mu^2 - \frac{64}{27}\lambda^2\mu^4 + \cdots ,\\
g_2 &= \frac{284}{81}\lambda\mu + \frac{2432}{243}\lambda^2\mu^3 + \cdots ,\\
g_3 &= \frac{4}{81}\lambda - \frac{1024}{81}\lambda^2\mu^2 + \cdots ,\\
g_4 &= \frac{2348}{729}\lambda^2\mu + \cdots ,
\end{aligned}
$$

which recover the displayed coefficients of Π_1 and Π_2.

6. An exact mechanism for all diagonals

The preceding identities are not isolated fits. Expand the polynomial in W as

$$WT_n(1+2W) = \sum_{r=0}^{n} C_{n,r}W^{n+1-r}, \tag{23}$$

where $C_{n,0} = 2^{2n-1}$ for $n \geq 1$. For each fixed r, the standard finite coefficient formula for T_n shows that $C_{n,r}/C_{n,0}$ is a polynomial in n on its natural range; for example

$$\frac{C_{n,1}}{C_{n,0}} = \frac{n}{2}, \qquad \frac{C_{n,2}}{C_{n,0}} = \frac{n(2n-3)}{16}.$$

Together with coefficient extraction from $a(t)^{n+1-r}$, this is the elementary reason the fixed-diagonal ratios below are polynomial in n. With $a(t)$ from (3), define

$$L_j(n) = \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{C_{n,r}}{C_{n,0}} \left(\frac{3}{4}\right)^r [t^{j-2r}]a(t)^{n+1-r}. \tag{24}$$

A direct coefficient extraction in (4) gives

$$\boxed{p_j(n) = (-1)^j (2n+1)^{\underline{j}} L_j(n),} \tag{25}$$

where $m^{\underline{j}} = m(m-1)\cdots(m-j+1)$. Formula (25), together with (18), is an algorithmic generator for the asymptotic filter triangle. For each fixed j, only a finite jet of $a(t)$ and finitely many large-W Chebyshev coefficients are required.

There are two useful structural consequences. First,

$$\boxed{\deg p_j = 2j.}$$

Indeed, the falling factorial in (25) has degree j. In $L_j(n)$, the r-th summand has degree at most $r + (j - 2r) = j - r$; the $r = 0$ term has degree exactly j, because $[t^j]a(t)^{n+1}$ has nonzero leading term $n^j/j!$. Hence L_j has degree j, and the product has degree $2j$.

Second, the falling factorial automatically enforces the natural lower range of the triangle. If $j > 2n+1$, then $(2n+1)^{\underline{j}} = 0$. Thus diagonals that do not yet exist for a given filter vanish without an additional convention; for example $p_4(1) = 0$, while the fourth downward diagonal first occurs at $n = 2$.

7. Interpretation and scope

The two orders of operation expose complementary local objects. Formal summation before residue extraction produces the geometric boundary resolvent (7), whose singularity occurs at $\lambda = 3/16$. Residue extraction first produces the entire master function (14) and its differential descendants. The difference is explained by the residue: the nth mode requires the $(2n+1)$st Taylor coefficient of e^{-xt}, introducing the factorial $(2n+1)!^{-1}$ that removes the geometric singularity.

Equivalently, set $r = 16\lambda/3$, so that $z = \mu\sqrt{r}$. The two orders act on the same unit coefficient sequence:

$$\sum_{n\geq 0} r^n = \frac{1}{1-r} \quad \longmapsto \quad \sum_{n\geq 0} \frac{(\mu^2 r)^n}{\Gamma(2n+2)} = E_{2,2}(\mu^2 r) = \frac{\sinh(\mu\sqrt{r})}{\mu\sqrt{r}}.$$

Thus the sum-first model is geometric, while the residue-first model is its Mittag–Leffler regularization. The critical coordinate is retained: $r = 1$, equivalently $\lambda = 3/16$, corresponds to $z = \mu$; what changes is its analytic expression, from a finite pole to entire hyperbolic growth. This is closely analogous to generalized Borel–Le Roy regularization, but conventions vary across that literature, whereas the canonical special function here is unambiguously $E_{2,2}$ [2].

We emphasize the limited claim. The calculation is a local asymptotic theory for the prime-filter family attached to this cubic Chebyshev trace. It does not establish the Riemann hypothesis, nor does it by itself give a new prime number theorem. Its content is the exact boundary scaling, the factorial regularization mechanism, and the resulting master-function/dilation calculus.

Conclusion

The calculation can be summarized by one contrast:

$$\boxed{\text{geometric resummation} \quad \longleftrightarrow \quad \text{factorially regularized residue calculus.}}$$

The collapsing Chebyshev radius near $s = 2$ is not featureless. At the natural scale $v \asymp (s-2)^2$ it resolves to a finite critical coordinate $\lambda_c = 3/16$. Respecting residue extraction before resummation reveals a second structure: the highest-degree edges of all prime filters balance simultaneously under $x = \mu/(s-2)$ and are generated by one hyperbolic master function. The local Laurent geometry of the cubic quotient determines polynomial operators in the filter index; the dilation correspondence $n \leftrightarrow (z/2)\partial_z$ turns these into differential descendants of $\sinh z/z$. In schematic form,

$$\boxed{q(s) \longrightarrow a(t) \longrightarrow p_j(n) \longrightarrow p_j(\mathcal{D}) \longrightarrow g_j(z).}$$

This gives a concise calculus for the asymptotic prime-filter triangle and a precise model of the noncommutativity between local residue extraction and global Chebyshev resummation.

The most elementary truth exposed by the calculation is perhaps also the most useful one. A singular generating family may carry substantially different analytic information before and after a local functional is applied, even when every finite mode is perfectly well defined. Here the residue at the cubic exceptional point changes the boundary layer itself: the geometric pole at $r = 1$ is replaced, in the residue-first scaling, by an entire hyperbolic function with factorially damped coefficients. The critical coordinate is not forgotten—at $r = 1$, $z = \mu$—but its analytic expression has changed from finite singularity to exponential-type growth. The regularized seed is precisely $E_{2,2}(z^2) = \sinh z/z$, and the dilation calculus records how it propagates through every fixed diagonal of the filter hierarchy.

This is a local statement, but it is a complete one at the level described here. Its value is not that it resolves the global problem from which the filters arose; it is that a seemingly singular corner of that problem has a simple internal law.

References

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