

Reducing the dimension of mathematical models near border-collision bifurcations

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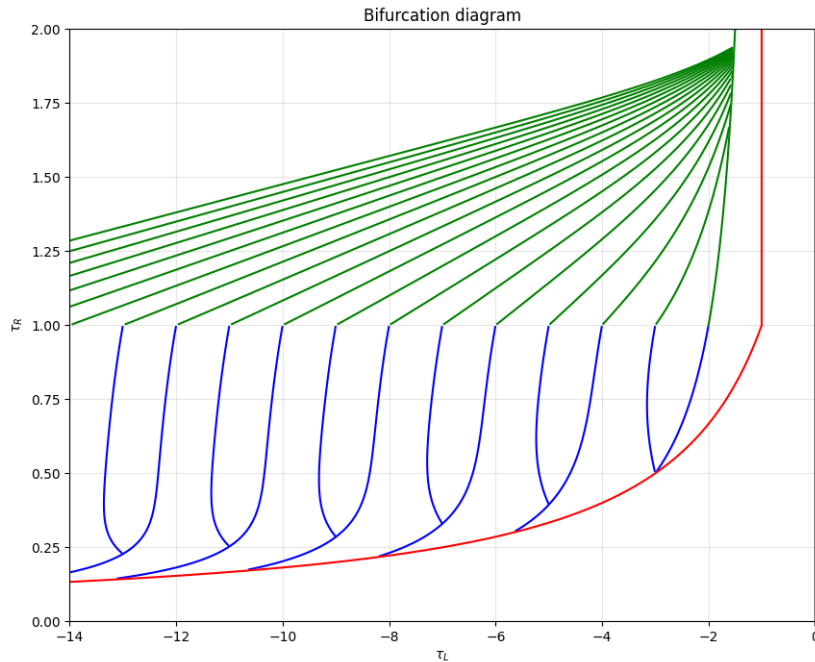
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1 Problem Statement

We are given some fixed instance of the Border Collision Normal Form, for constant parameters $\tau_L, \tau_R \in \mathbb{R}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} \begin{pmatrix} \tau_L x + y - 1 \\ 0 \end{pmatrix}, & \text{if } x \leq 0 \\ \begin{pmatrix} \tau_R x + y - 1 \\ (1 - \tau_R)x \end{pmatrix}, & \text{if } x \geq 0 \end{cases}$$

Our objective is to analyze the long time dynamics of this system and identify bifurcation curves in the parameter space, with particular emphasis the structure of chaotic attractors. Before presenting the detailed derivations of the bifurcation curves, we first display the bifurcation diagram in order to guide the reader better:



2 Period Two Orbits

We consider period-two orbits consisting of one iteration under f_L followed by one under f_R . Such an orbit $\{P_1, P_2\}$ exists provided $\tau_L \neq 1$ and $\tau_R \neq 0$, and is given explicitly via

$$P_1 = \left(\frac{2}{\tau_L - 1}, \frac{(1 - \tau_R)(\tau_L + 1)}{\tau_R(\tau_L - 1)} \right), \quad P_2 = \left(\frac{\tau_L + 1}{\tau_R(\tau_L - 1)}, 0 \right)$$

Admissibility of the orbit requires P_1 to lie in the left half plane and P_2 in the right half plane, which implies $\tau_L < 1$ together with one of the following parameter conditions:

$$\begin{cases} \tau_L \leq -1, & \tau_R > 0, \\ -1 \leq \tau_L < 1, & \tau_R < 0 \end{cases}$$

The stability is determined by the eigenvalues of the Jacobian of $f_R \circ f_L$, which is 0 and $\tau_L \tau_R + 1 - \tau_R$. The orbit is asymptotically stable if and only if

$$|\tau_L \tau_R + 1 - \tau_R| < 1$$

This condition $\tau_L < 1$ defines the bifurcation curve separating stable period-two orbits from a single point. Our first bifurcation line is at $\tau_L = 1$. This has not been included in the bifurcation diagram (yet). By our second condition, the boundary between stable period-two orbits and chaotic attractors is given by

$$\tau_R = -\frac{2}{\tau_L - 1}$$

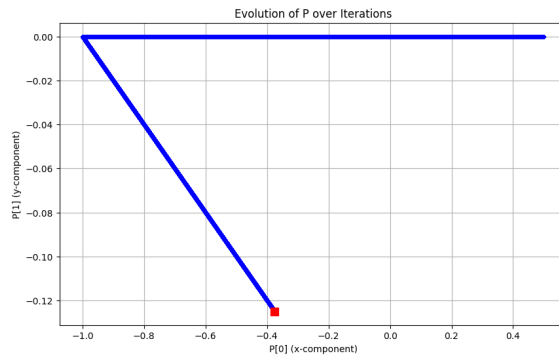
However, it is only relevant where the period-two orbit exists and is admissible. Since the "stability curve" lies entirely in $\tau_R < 0$, it intersects the admissible region only at

$$-1 \leq \tau_L < 1$$

Therefore, the curve ends at $\tau_L = -1$. This key bifurcation curve is presented in red. At $\tau_L = 1$, the formula for this period-two orbit breaks down. Beyond this point, $\tau_L \geq 1$, period-two orbits no longer exist and all iterations collapse onto a single stable fixed point.

3 Chaotic Attractors for $\tau_R > 1$

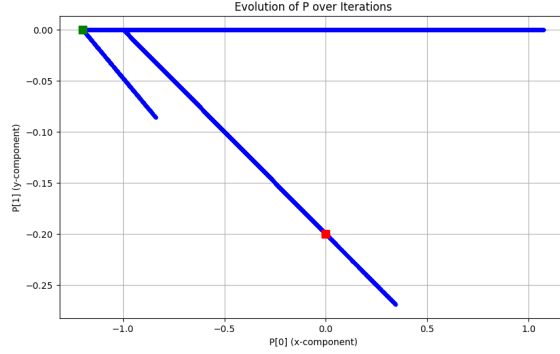
Before we move on, let's consider the simplest type of chaotic attractor. This is an example of where $\tau_L = -1.5$ and $\tau_R = 1.25$:



This is an attractor that resembles an upside down L shape. Here there are two segments. We find the endpoint by

$$(f_R \circ f_L)(-1, 0) = \begin{pmatrix} -\tau_L \tau_R - \tau_R - 1 \\ -(1 - \tau_R)(\tau_L + 1) \end{pmatrix}$$

as outlined by the red point on the attractor. Now for the values $\tau_L = -1.73$ and $\tau_R = 1.25$ can be seen:



Now we have an LL -shaped attractor, with 3 segments. We wish to discern when this topological change occurs. Indeed, this happens when the endpoint of the L -attractor crosses the switching manifold, and a kink is formed on the leftmost side. Therefore we look where

$$-\tau_L \tau_R - \tau_R - 1 = 0 \quad \implies \quad \tau_R = \frac{1}{-\tau_L - 1}$$

which gives us the right-most green curve in our bifurcation diagram. Now to find the transition point from an LL -attractor to an LLL -attractor, we wish to find the endpoint of the leftmost segment of this LL -attractor, and see where it crosses the y -axis. To find this endpoint, we would need to find the leftmost point of the attractor L_2 . This point L_2 is the point on the second segment with $x = 0$.

We want a point $(x, 0)$ on the x -axis such that after one iteration of f_R the x -coordinate becomes zero:

$$f_R(x, 0) = (0, \text{something}).$$

Solving gives

$$x = \frac{1}{\tau_R}$$

and indeed

$$f_R\left(\frac{1}{\tau_R}, 0\right) = \left(0, \frac{1 - \tau_R}{\tau_R}\right)$$

The point $(0, \frac{1 - \tau_R}{\tau_R})$ is presented in the figure in red. Now the leftmost-point

$$L_2 = f_L\left(0, \frac{1 - \tau_R}{\tau_R}\right) = \left(\frac{1 - 2\tau_R}{\tau_R}, 0\right)$$

is presented in green. Now we take

$$(f_R^2 \circ f_L) \left(\frac{1-2\tau_R}{\tau_R}, 0 \right) = \left(\begin{array}{c} \tau_R(\tau_L(1-2\tau_R) - \tau_R - 1) + \frac{(1-\tau_R)(\tau_L(1-2\tau_R) - \tau_R)}{\tau_R} - 1 \\ (1-\tau_R)(\tau_L(1-2\tau_R) - \tau_R - 1) \end{array} \right)$$

To get the coordinates for the endpoint of the leftmost segment. Plotting the contour

$$\tau_R(\tau_L(1-2\tau_R) - \tau_R - 1) + \frac{(1-\tau_R)(\tau_L(1-2\tau_R) - \tau_R)}{\tau_R} - 1 = 0$$

gives the second-left-most green bifurcation curve in the diagram. Indeed this marks the curve where the LL -attractors are separated from the LLL -attractors.

3.1 General case for chaotic attractors of type L^n

We now plot the bifurcation curves for the general case of an L^n shaped attractor with $n + 1$ prongs. This process in steps looks like follows:

1. Find some point $(x^n, 0)$ on the x -axis such that $f_R^{n-1}(x^n, 0)$ lands on the switching manifold. Label this final point $(0, y^n)$.
2. Compute $f_L(0, y^n) = f_R(0, y^n)$ which gives us the left most point L^n of the attractor.
3. Compute $f_R^n \circ f_L(L)$ to get the coordinates for the endpoint of the left-most segment labeled (x_n, y_n) .
4. Take the x -coordinate of the endpoint of the left-most segment and plot the contour $x_n = 0$.

The second and third steps can be put together in one go, we presented as above for clarity. For the first step we first represent f_R as an affine linearisation or something:

$$f_R \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \tau_R & 1 \\ 1 - \tau_R & 0 \end{bmatrix}}_{A_R} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_B$$

So we can see

$$f_R^{n-1} \begin{bmatrix} x^n \\ 0 \end{bmatrix} = A_R^{n-1} \begin{bmatrix} x^n \\ 0 \end{bmatrix} + (I + A_R + \dots + A_R^{n-2}) \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

So by defining $C = \sum_{i=0}^{n-2} A_R^i B$ then clearly

$$x^n = -\frac{[C]_1}{[A_R^{n-1}]_{11}}$$

now substituting x^n to find:

$$y^n = [A_R^{n-1}]x^n + [C]_2$$

which gives

$$y^n = \left[\sum_{i=0}^{n-2} A_R^i B \right]_2 - \frac{[A_R^{n-1}]_{21}}{[A_R^{n-1}]_{11}} \left[\sum_{i=0}^{n-2} A_R^i B \right]_1$$

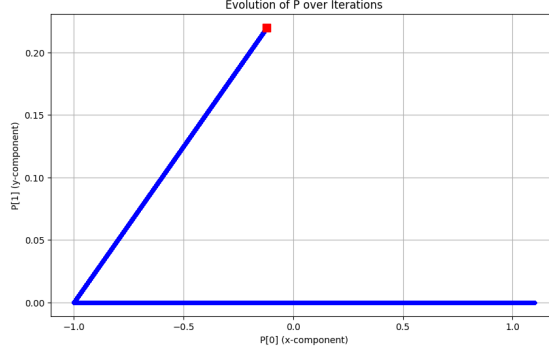


Figure 1: $\tau_L = -2.1$ $\tau_R = .8$

Now the leftmost point of the attractor is simply

$$L^n := f_L(0, y^n) = [y^n - 1 \quad 0]^T$$

and applying f_L one more time gives

$$f_L(L^n) = \left(\tau_L(y^n - 1) - 1, 0 \right)$$

Which allows us to find the endpoint

$$f_R^n \circ f_L^2(0, y^n) = f_R^n \begin{bmatrix} \tau_L(y^n - 1) - 1 \\ 0 \end{bmatrix} = A_R^n \begin{bmatrix} \tau_L(y^n - 1) - 1 \\ 0 \end{bmatrix} + \sum_{i=0}^{n-1} A_R^i B$$

Now given $D = \sum_{i=0}^{n-1} A_R^i B$ we are able to get the endpoints as follows:

$$\begin{aligned} x_n &= [A_R^n]_{11} \left(\tau_L(y^n - 1) - 1 \right) + [D]_1 \\ y_n &= [A_R^n]_{21} \left(\tau_L(y^n - 1) - 1 \right) + [D]_2 \end{aligned}$$

Each of the green bifurcation curves plot the contour

$$x_n = [A_R^n]_{11} \left(\tau_L \left(\left[\sum_{i=0}^{n-2} A_R^i B \right]_2 - \frac{[A_R^{n-1}]_{21}}{[A_R^{n-1}]_{11}} \left[\sum_{i=0}^{n-2} A_R^i B \right]_1 - 1 \right) - 1 \right) + \left[\sum_{i=0}^{n-1} A_R^i B \right]_1 = 0$$

4 Chaotic attractors with $\tau_R < 1$

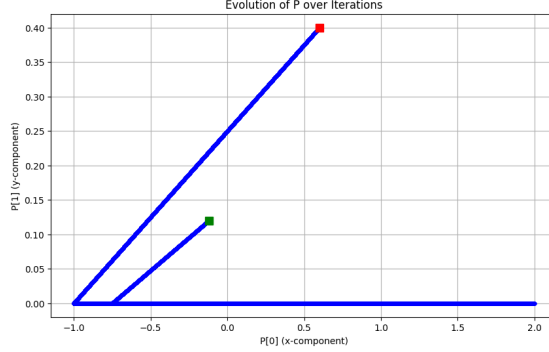
At $\tau_R = 1$ the system transitions from some expanding solution to contracting attractor as the Jacobian determinant $\det(A_R) = \tau_R - 1$ vanishes. A prototypical but simple example is presented in figure 1. Here we have an inverse L shaped attractor which we denote L^{-1} and has two segments. The right-most point of the attractor is simply the forward iterate under f_L of $(-1, 0)$:

$$f_L(-1, 0) = (-\tau_L - 1, 0)$$

The endpoint of the next segment is given by

$$f_R(-\tau_L - 1, 0) = \left(\tau_R(-\tau_L - 1) - 1, (1 - \tau_R)(-\tau_L - 1) \right)$$

as highlighted in the figure as the red point. As in the $\tau_R > 1$ case, when the endpoint of this segment crosses the switching manifold a kink is formed inside the subset $\{(x, 0) \in \mathbb{R}^2 : -1 < x < 0\}$ on the bottom segment. Such a case is when $\tau_L = -3.3, \tau_L = .8$ as pictured below:



One can quickly confirm as the x -coordinate of $f_R \circ f_L(-1, 0)$ crosses 0 we get a chaotic structure that is topologically distinct from the previous one. Indeed, the contour

$$\tau_R(-\tau_L - 1) - 1 = 0$$

is the right most blue curve in our bifurcation diagram. The general case is simple. We label the rightmost point $R = (-\tau_L - 1, 0)$ as the critical point of this family of attractors. The endpoint of the final segment of the $L^{-(n+1)}$ attractor is given by $f_R^n(R)$. Seeing our recursive map

$$\begin{aligned} x_k &= \tau_R x_{k-1} + y_{k-1} - 1 \\ y_k &= (1 - \tau_R)x_{k-1} \end{aligned}$$

can be reduced to

$$x_k = \tau_R x_{k-1} + (1 - \tau_R)x_{k-2} - 1$$

which is a second-order linear non-homogeneous recurrence relation which can be solved:

$$x_n = \frac{1 - (\tau_L + 2)(2 - \tau_R)}{(2 - \tau_R)^2} + \left[\frac{(\tau_L(\tau_R - 1) + \tau_R)(2 - \tau_R) - 1}{(2 - \tau_R)^2} \right] (\tau_R - 1)^n - \frac{n}{2 - \tau_R}$$

Each of the blue bifurcation curves in the diagram map the contours $x_n = 0$.

5 Gaps in L -attractors for $\tau_R < 1$

For the L -shaped attractors with $\tau_R < 1$ gaps might appear in the segments. Such an example is shown in Figure 2. In this simple case, a single gap appears in the bottom segment. To find when this gap occurs we find the point on the left and right side of the gap, which we label G_L and G_R respectively. The plot above has a red point to represent G_L and a green point to represent G_R . In this case we find

$$G_R = (f_L \circ f_R \circ f_L)(-1, 0) = (-\tau_L^2 \tau_R - 2\tau_L + \tau_R - 2, 0)$$

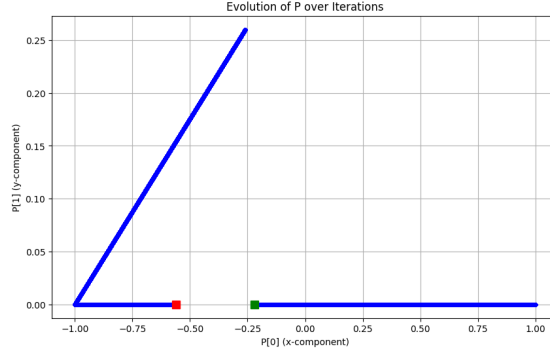


Figure 2: $\tau_L = -2$ $\tau_R = .74$

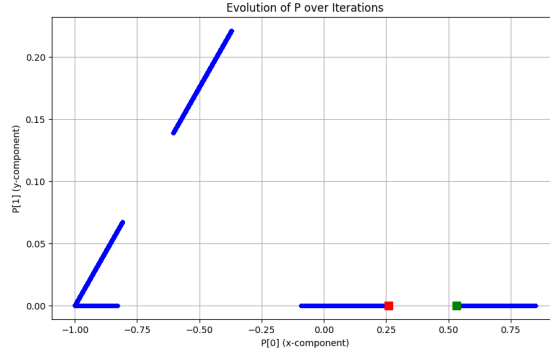


Figure 3: $\tau_L = -1.85$ $\tau_R = .74$

$$G_L = f_L(G_R) = (f_L^2 \circ f_R \circ f_L)(-1, 0) = (-\tau_L^3 \tau_R - 2\tau_L^2 + \tau_L \tau_R - 2\tau_L - 1, 0)$$

Now to find when an L attractor transforms into an L -attractor with a gap, we simply plot the curve $[G_L]_1 = [G_R]_1$. Indeed, we plot the bifurcation curve in the relevant region of parameter space:

$$\tau_R = \frac{2\tau_L^2 - 1}{\tau_L^2 - \tau_L^3 + \tau_L - 1}$$

The only other case in which there is another gap-variant of the L -attractor is when there is one gap on the upper segment, and two gaps on the lower segment. Such an example is in Figure 3. Because this is unique and distinct to the L -attractor with one gap, we only need to keep track of the second gap on the bottom segment (or alternatively the single gap on the upper segment). We use the same notation as before:

$$G_R = (f_L^3 \circ f_R \circ f_L)(-1, 0) = (\tau_L^2(-\tau_L^2 \tau_R - 2\tau_L + \tau_R - 2) - \tau_L - 1, 0)$$

$$G_L = (f_L \circ f_R)(G_R) = (\tau_L(\tau_R[G_R]_1 - 1) + (1 - \tau_R)[G_R]_1 - 1, 0)$$

We plot this $[G_L]_1 = [G_R]_1$ as a quadratic contour:

$$\tau_R^2 [(\tau_L^2 - \tau_L^4)(\tau_L - 1)] - \tau_R [(2\tau_L^3 + 2\tau_L^2 + \tau_L + 1)(\tau_L - 1)] - (\tau_L + 1) = 0$$

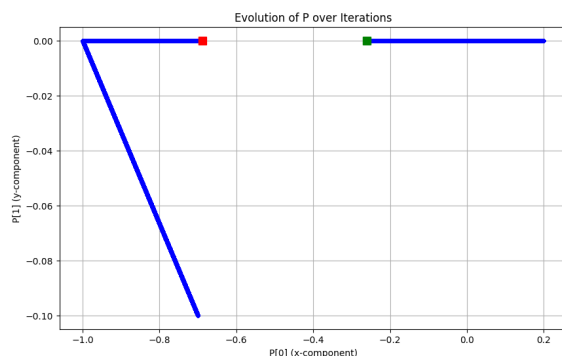
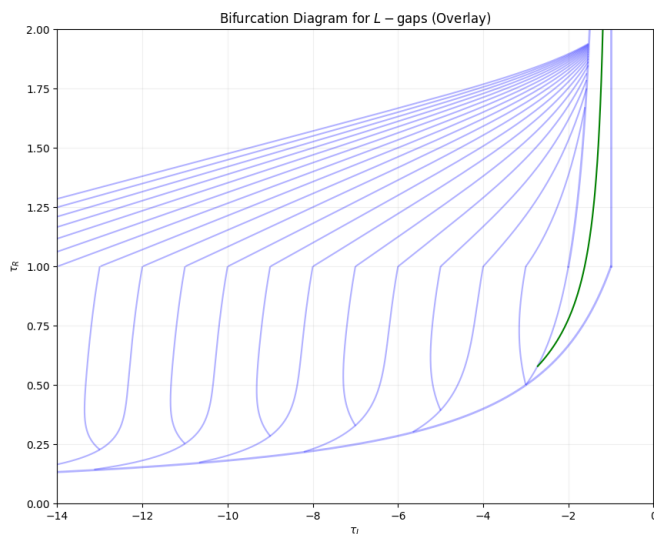


Figure 4: $\tau_L = -1.2$ $\tau_R = 1.5$

6 Gaps in L -attractors for $\tau_R > 1$

This case when a single gap appears for $\tau_R > 1$ is precisely the same as the previous case. Here, the attractor is presented in figure 4.

Because there is no difference in the formula for G_L and G_R in this case, the bifurcation curve for gaps in the L -attractor for $\tau_R > 1$ is the same as in the case $\tau_R < 1$. We take the previous bifurcation diagram presented, and overlay this curve separating the L -attractors from the L -attractors with a single gap.



Now for the final case, we may have a gap appear as presented in figure 5. Here the endpoint of the left side of the second gap in the first segment G_L is highlighted in red, and opposite is G_R highlighted in green. Here the formulas for G_L and G_R are precisely the same as in the previous case, where we examined L -attractors with two gaps on the bottom segment. We plot the quadratic contour and overlay it on our bifurcation diagram to get figure 6. Now there is no reason why these gaps should disappear when the attractors transition from L^{-1} to L^{-2} .

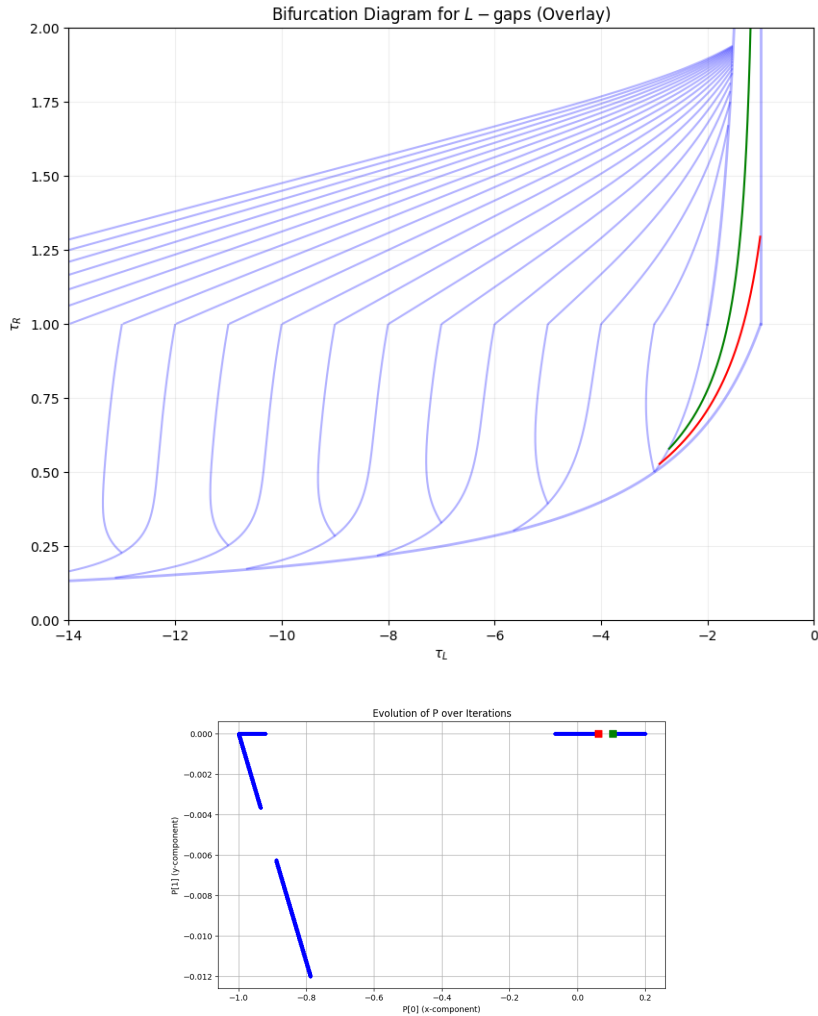


Figure 5: $\tau_L = -1.2$ $\tau_R = 1.06$

Computational plotting confirms this conjecture, while it is incredibly difficult to find such cases, and they are so skinny that plotting them renders a single line.

7 Final Question

It remains to rigorously prove what we have outline in this report. The first step would be to prove for some region in parameter space we get the L -attractor. Proving this would involve showing the image on the left-segment is mapped to a restriction of the x -axis, and that the image of the x -axis is confined to the left-most ray emanating from the critical point $(-1, 0)$ to $(f_R \circ f_L)(-1, 0)$. One would then show for any (τ_L, τ_R) in this specific region and initial seed P_0 the forward orbit of P_0 under f eventually lands on one of these two segments.